Brush-Up Maths for Data Science (2025)

Lecture Slides, Aug. 24th

Nicklas S. Andersen

University of Southern Denmark (SDU)
Department of Mathematics & Computer Science (IMADA)

Multivariable Functions

- Building Intuition

We have focused on functions of a single variable:

- Each input is a single number x
- Each output is a single number $f(x)$

Many situations, however, involve relationships between more than one independent variable.

Examples:

- Temperature:
  - depends on latitude and longitude
- Company profit:
  - depends on units sold and unit price
- Crop yield:
  - depends on fertilizer, rainfall, soil pH, etc.

Domains of (real-valued) functions:

- Single-variable functions $f(x)$
  - Domain: interval(s) on the number line $\mathbb{R}$
- Two-variable functions $f(x_1, x_2)$
  - Domain: region(s) in the plane $\mathbb{R}^2$
- Three-variable functions $f(x_1, x_2, x_3)$
  - Domain: region(s) in space $\mathbb{R}^3$
- n-variable functions $f(x_1, x_2, \ldots, x_n)$
  - Domain: region(s) in $\mathbb{R}^n$

As we add more variables, the domain lives in higher-dimensional spaces.

A multivariable function maps points located in these higher-dimensional spaces to the real numbers $\mathbb{R}$.

Multivariable Functions

- Definition

More formally, a real-valued function of n variables is a rule that assigns to each input:

$$(x_1, x_2, \ldots, x_n) \in D \subseteq \mathbb{R}^n$$

exactly one real number:

$$f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$$

This is written as:

$$f : D \to \mathbb{R}$$

The range (or image) of f is the set of all actual outputs:

$$\text{Range}(f) = \{f(x_1, x_2, \ldots, x_n) \in \mathbb{R} \mid (x_1, x_2, \ldots, x_n) \in D\}$$

Multivariable Functions

- Visualizing Functions of Multiple Variables

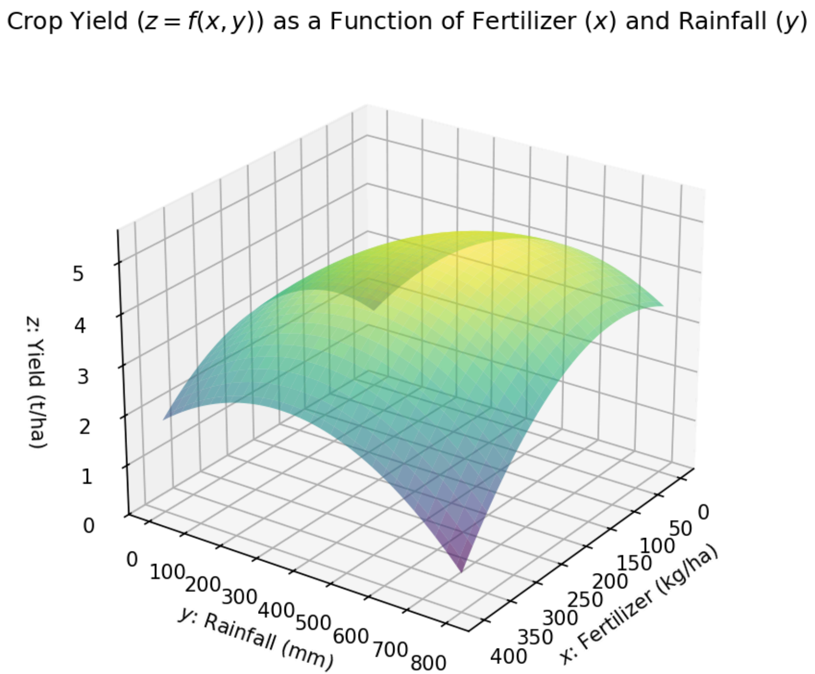
Note that, when we want to visualize:

- A single-variable function, then:
  - The graph of $f(x)$ lies in $\mathbb{R}^2$
  - A plane: input axis + output axis
- two-variable function, then:
  - The graph of $f(x_1, x_2)$ lies in $\mathbb{R}^3$
  - A 3D space: two input axes + output axis
- A function depending on three or more variables:
  - The graph of $f(x_1, x_2, \ldots, x_n)$ lies in $\mathbb{R}^{n+1}$

We can not visualize such functions directly.

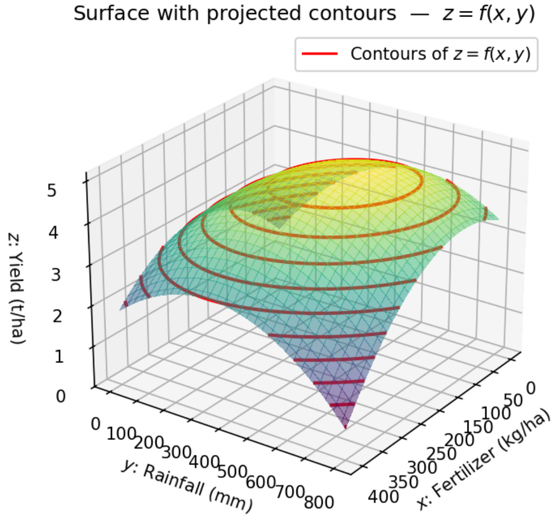
We need projections, level sets, and slices to study the behavior.

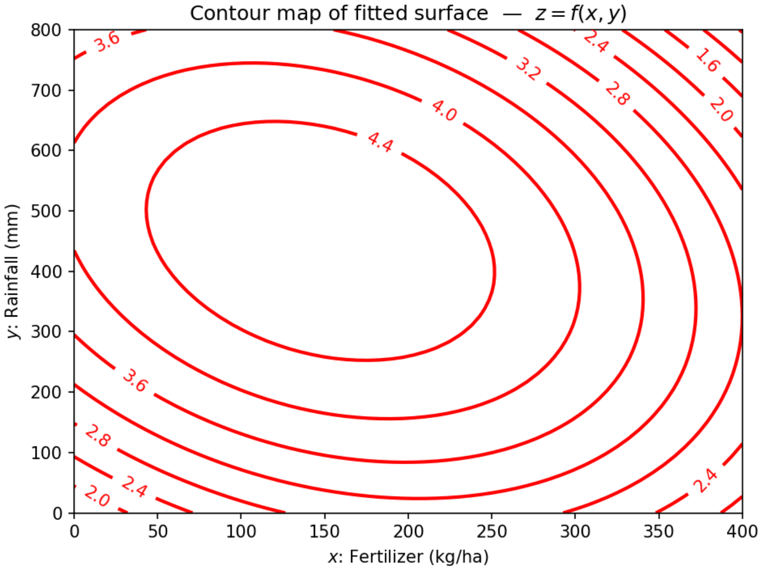
To illustrate these techniques, we extend the crop yield model we studied earlier to include additional factors:



Level Curves/Contours

- Definition





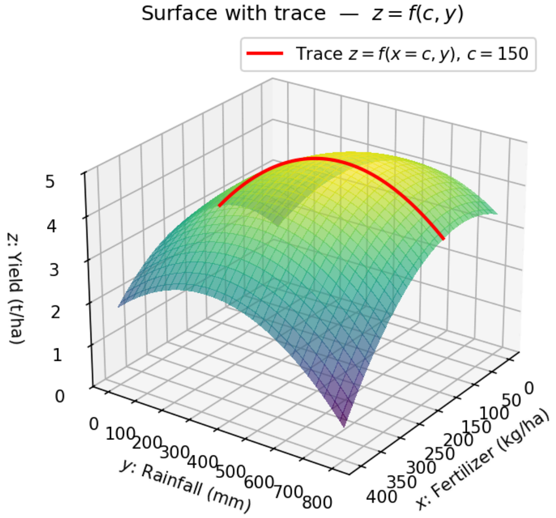
For a function of two variables $f(x, y)$, a level curve (or contour) is the set of all points (x, y) in the domain where the function takes a fixed constant value k:

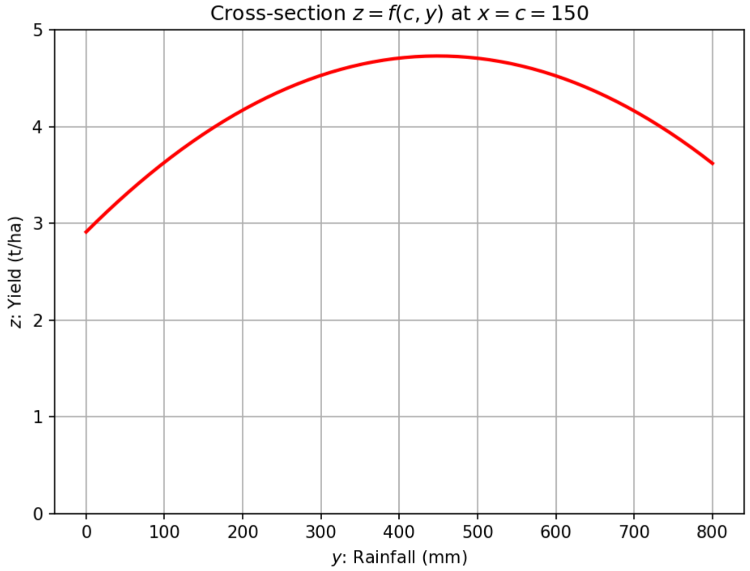
$$f(x, y) = k$$

In the xy-plane, a level curve connects all points where f produces the same output.

Function Traces

- Definition





Function traces enable us to examine cross-sections of the surface by fixing one variable and varying the other.

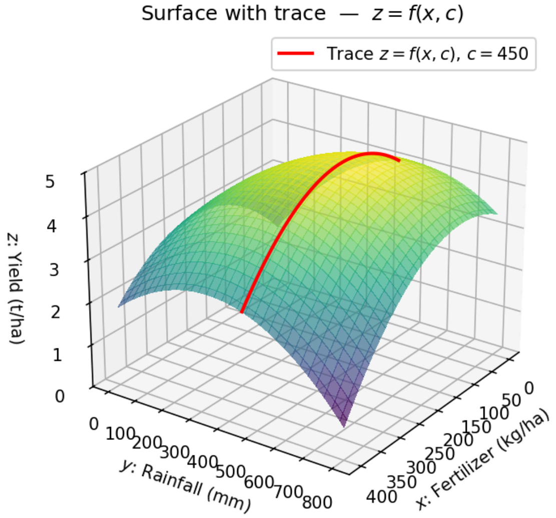
For $f(x, y)$, a trace is obtained by fixing one variable and letting the other vary (trace in the x-direction):

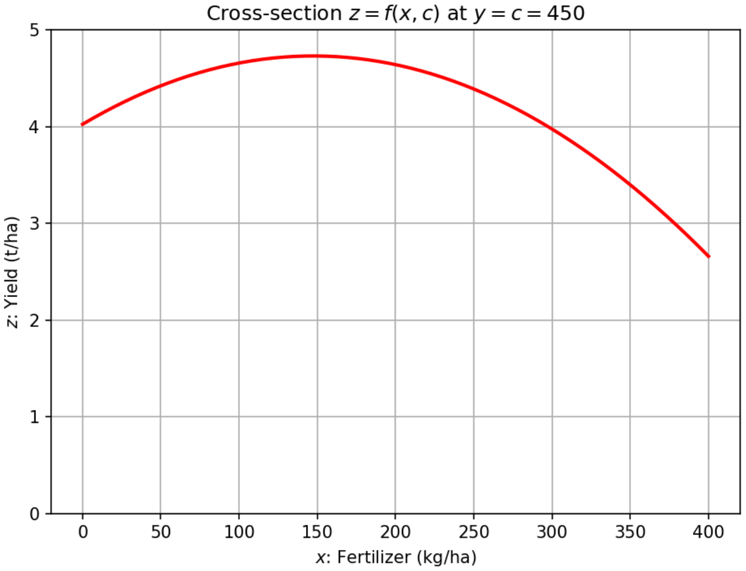
$$z = f(x, c)$$

This curve lies in the vertical plane parallel to the xz-plane.

Function Traces

- Definition





Function traces enable us to examine cross-sections of the surface by fixing one variable and varying the other.

For $f(x, y)$, a trace is obtained by fixing one variable and letting the other vary (trace in the y-direction):

$$z = f(c, y)$$

This curve lies in the vertical plane parallel to the yz-plane.