



Brush-Up Maths for Data Science (2025)

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Derivatives

What is a derivative?

- A derivative tells us the slope of the function at a single point on its graph

Central to the concept of a derivative is the concept of:

- Secant lines (slopes over an interval)
- Tangent lines (slopes at a single point)

These ideas allow us to quantify how a function changes.

Secant Lines

Suppose that we have:

- A function f
- A point a
- Another point $a + h$ nearby ($h \neq 0$)

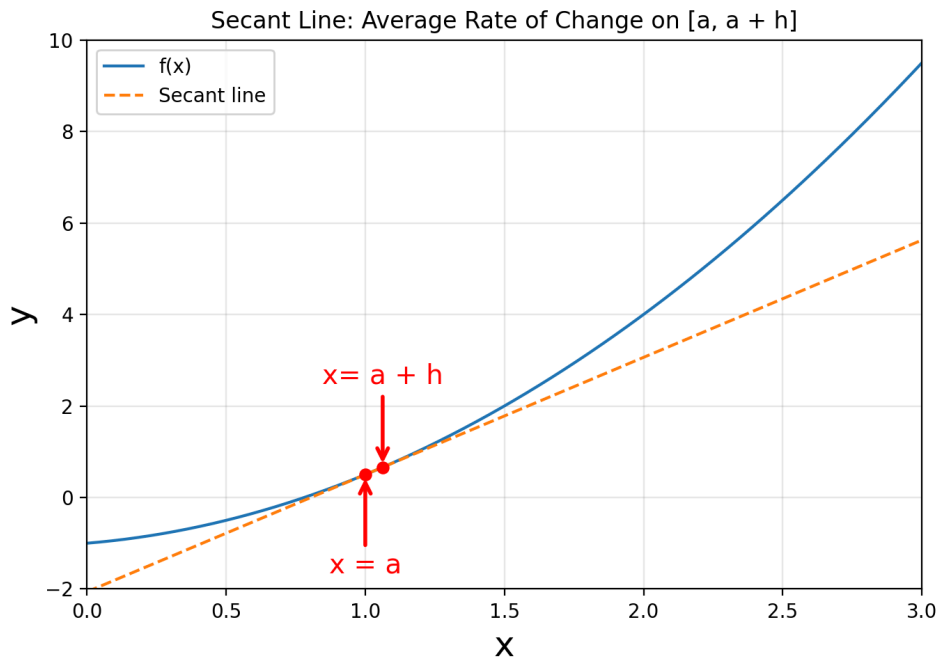
In this case, the slope of the secant line between

$$(a, f(a)) \quad \text{and} \quad (a + h, f(a + h))$$

represents the average rate of change of f on the interval $[a, a + h]$.

The slope of this line is:

$$m_{\text{sec}} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}$$



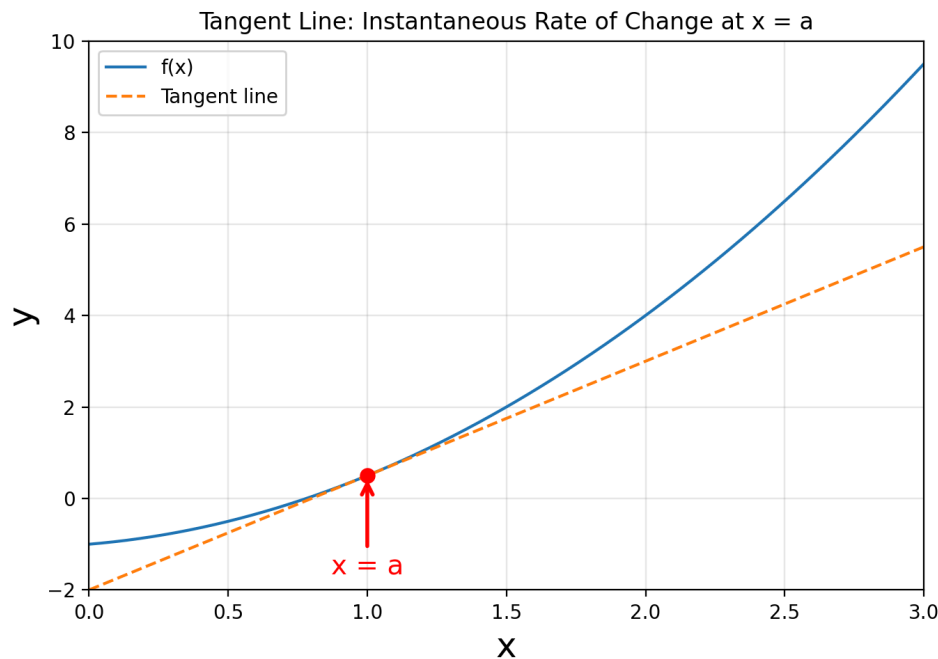
As h becomes very small, the secant line begins to "approach" the tangent line at a .

Tangent Lines

The tangent line is the "limit position" of the secant line as the second point approaches the first:

$$m_{\text{tan}} = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit actually exists.



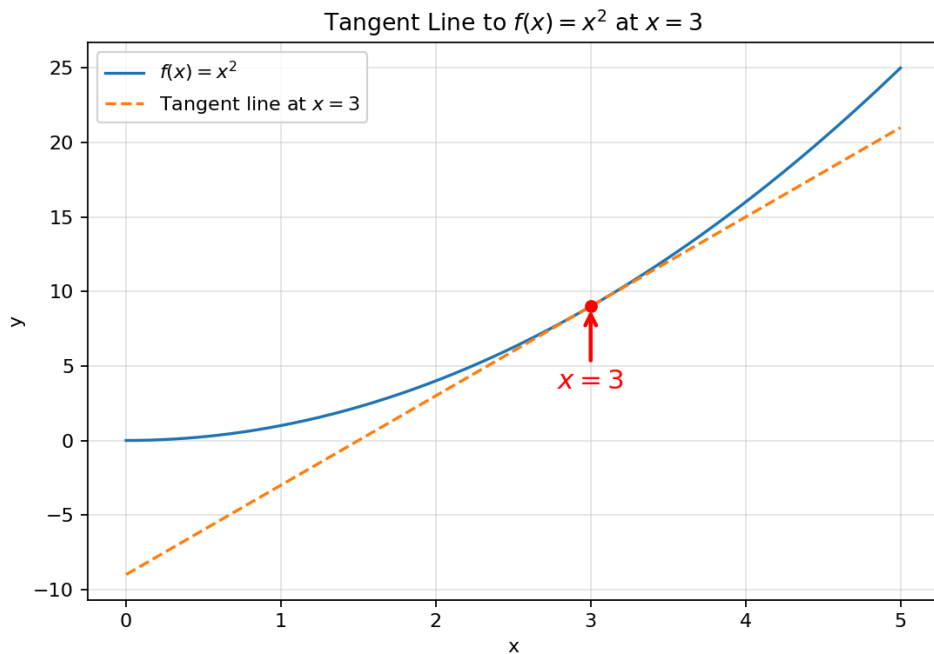
As h becomes very small, the secant line begins to "approach" the tangent line at a .

Tangent Lines

- Example

Find the slope of the tangent line to the graph of $f(x) = x^2$ at the point $x = 3$.

$$\begin{aligned}m_{tan} &= \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h} \\&= \lim_{h \rightarrow 0} \frac{(3+h)^2 - 9}{h} \\&= \lim_{h \rightarrow 0} \frac{9 + 6h + h^2 - 9}{h} \\&= \lim_{h \rightarrow 0} \frac{h(6+h)}{h} \\&= \lim_{h \rightarrow 0} 6 + h \\&= 6\end{aligned}$$



The Derivative of a Function

- Definition

To define the derivative of a function, let:

- $D \subseteq \mathbb{R}$ be the domain of f
- $f : D \rightarrow \mathbb{R}$ be a function
- $a \in D$ where values of f are defined close to a

The derivative of f at a is then defined as:

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

Note that:

- If $f'(a)$ exists, we say f is differentiable at a .
- If $f'(x)$ exists for every $x \in D$, we say f is differentiable on D .

Instead of writing $f'(x)$ we sometimes write:

$$\frac{df}{dx} \quad \text{or} \quad \frac{d}{dx} f(x)$$

The Derivative of a Function

- Example: Linear Functions

Consider the linear function:

$$f(x) = ax + b, \quad a \neq 0, \quad b \in \mathbb{R},$$

for any $x \in \mathbb{R}$.

We compute the derivative of the function, as follows:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(a(x+h) + b) - (ax + b)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(ax + ah + b) - (ax + b)}{h} \\ &= \lim_{h \rightarrow 0} \frac{ax + ah + b - ax - b}{h} \end{aligned}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{ah}{h} \\ &= \lim_{h \rightarrow 0} a \\ &= a \end{aligned}$$

Intuitively, this makes sense, as a linear function is a straight line with constant slope.

The Derivative of a Function

- Example: Quadratic Functions

Similarly, consider the quadratic function:

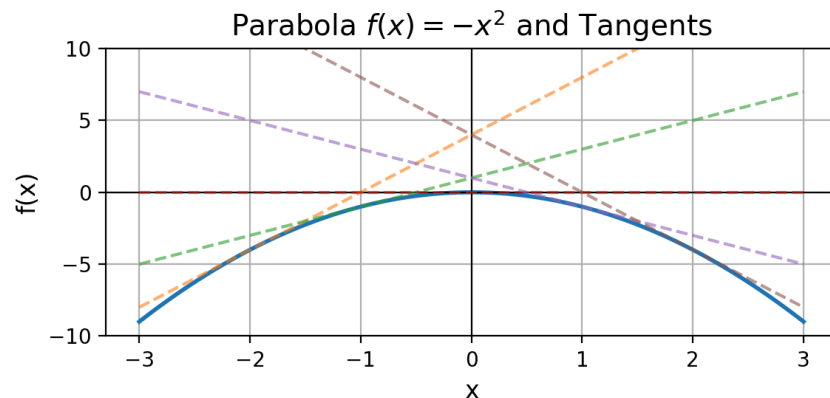
$$f(x) = ax^2, \quad a \neq 0,$$

for any $x \in \mathbb{R}$.

We compute the derivative of the function, as follows:

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{a(x+h)^2 - ax^2}{h} \\ &= a \cdot \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= a \cdot \lim_{h \rightarrow 0} \frac{x^2 + h^2 + 2hx - x^2}{h} \\ &= a \cdot \lim_{h \rightarrow 0} \frac{h^2 + 2hx}{h} \end{aligned}$$

$$\begin{aligned} &= a \cdot \lim_{h \rightarrow 0} \frac{h(h + 2x)}{h} \\ &= a \cdot \lim_{h \rightarrow 0} h + 2x \\ &= 2ax \end{aligned}$$



- If $a \neq 0$ and $x > 0$ the slope is negative
- If $a \neq 0$ and $x < 0$ the slope is positive

Some Common Derivatives

Below is a table of some of the most frequently used derivatives. Here $k, a \in \mathbb{R}$, $a > 0$, and $n \neq 0$.

Function $f(x)$	Derivative $f'(x)$	Notes
k	0	Constant rule
x^n	nx^{n-1}	Power rule
$\ln(x)$	$\frac{1}{x}$	$x > 0$
e^x	e^x	—
a^x	$\ln(a) \cdot a^x$	$a > 0$
$\sin(x)$	$\cos(x)$	—
$\cos(x)$	$-\sin(x)$	—

Common Differentiation Rules

Just like for limits, there are certain rules that we can apply when differentiating functions. In this context, let f and g be differentiable functions on an interval, with $g(x) \neq 0$. The following rules then hold:

Operation

Formula

Constant Multiple

$$\frac{d}{dx}(kf(x)) = kf'(x), \quad k \in \mathbb{R}$$

Sum

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

Difference

$$\frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x)$$

Product

$$\frac{d}{dx}(f(x) \cdot g(x)) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

Quotient

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}, \quad g(x) \neq 0$$

Common Differentiation Rules

- Examples

Differentiate the functions:

- $f(x) = 2x^2$
- $g(x) = x$

Then consider the derivative of:

- The product $f(x) \cdot g(x)$
- The quotient $\frac{f(x)}{g(x)}$

We get from the table of derivatives that:

$$f'(x) = \frac{d}{dx}(2x^2) = 4x$$

$$g'(x) = \frac{d}{dx}(x) = 1$$

Now, computing the product, we get:

$$\begin{aligned}\frac{d}{dx}(f(x) \cdot g(x)) &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\ &= 4x \cdot x + 2x^2 \cdot 1 \\ &= 6x^2\end{aligned}$$

For the quotient, assume $x \neq 0$, we get:

$$\begin{aligned}\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) &= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2} \\ &= \frac{4x \cdot x - 2x^2 \cdot 1}{x^2} \\ &= \frac{2x^2}{x^2} \\ &= 2\end{aligned}$$

Common Differentiation Rules

- Examples (Continued)

Compute the derivative of the function:

$$h(x) = e^x \cdot \sin(x)$$

Identifying constituent parts:

- $f(x) = e^x$
- $g(x) = \sin(x)$

In the product:

$$h(x) = f(x) \cdot g(x)$$

We notice that this is a product of functions.

This means that we can use the product rule to compute the derivative of $h(x)$.

The product rule gives us:

$$\begin{aligned} h'(x) &= \frac{d}{dx}((f(x) \cdot g(x))) \\ &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \end{aligned}$$

From the table of derivatives we get that:

- $f'(x) = e^x$
- $g'(x) = \cos(x)$

Substituting in our results we get:

$$\begin{aligned} h'(x) &= e^x \cdot \sin(x) + e^x \cdot \cos(x) \\ &= e^x \cdot (\sin(x) + \cos(x)) \end{aligned}$$

Exercise Set

- Part 1

Using the limit definition:

1. Find the slope of the tangent line to the graph of $f(x) = x^3$ at $x = -1$.
2. Find the derivative of the function $f(x) = x^2 - 2x + 1$.

Differentiate the following functions using the derivatives and differentiation rules from the earlier tables:

3. $f(x) = 2x^5 + 7$

4. $f(x) = \frac{6}{x^2}$

5. $f(x) = \frac{3x+1}{4x-3}$

6. $f(t) = t \cdot e^t$

The Chain Rule

What is the chain rule?

- It is a rule for finding the derivative of the composition of two or more functions

Let f and g be functions such that:

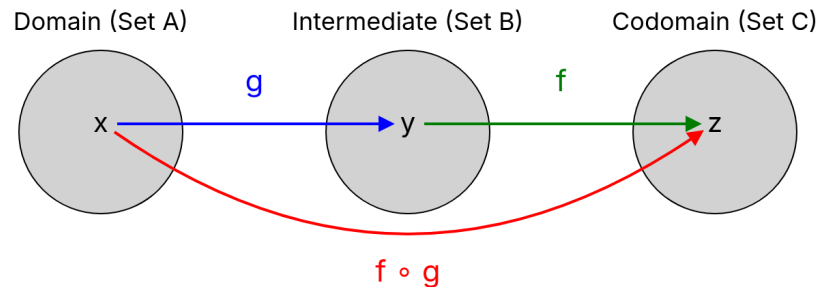
- g is differentiable at x
- f is differentiable at $g(x)$

For the composite function:

$$h(x) = (f \circ g)(x) = f(g(x))$$

the derivative is then defined as:

$$\frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x)$$



To differentiate $h(x) = f(g(x))$, follow the steps:

- Identify outer function f and inner function g
- Differentiate f with respect to its argument u to get $f'(u)$
- Substitute $u = g(x)$ into $f'(u)$ to obtain $f'(g(x))$
- Differentiate g with respect to its argument x to get $g'(x)$
- Multiply them to find:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

The Chain Rule

- Examples

Differentiate $h(x) = (\sin(x))^2$.

To do so, we let:

- The outer function be

$$f(u) = u^2, \text{ so } f'(u) = 2u$$

- The inner function be

$$g(x) = \sin(x), \text{ so } g'(x) = \cos(x)$$

Applying the Chain Rule, we then get:

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= 2 \sin(x) \cdot \cos(x) \end{aligned}$$

Differentiate $h(x) = e^{x^3+1}$.

To do so, we let:

- The outer function be

$$f(u) = e^u, \text{ so } f'(u) = e^u$$

- The inner function be

$$g(x) = x^3 + 1, \text{ so } g'(x) = 3x^2$$

Applying the Chain Rule, we then get:

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= e^{x^3+1} \cdot 3x^2 \end{aligned}$$

Derivatives & Local Extrema

The derivative $f'(x)$ represents the slope of the tangent line at the point $(x, f(x))$.

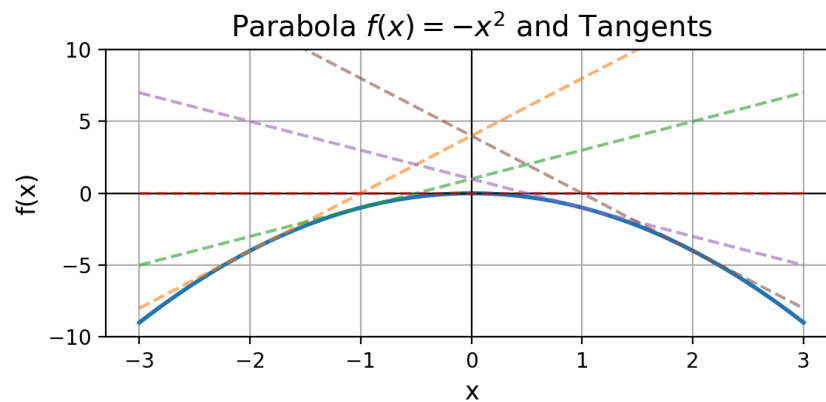
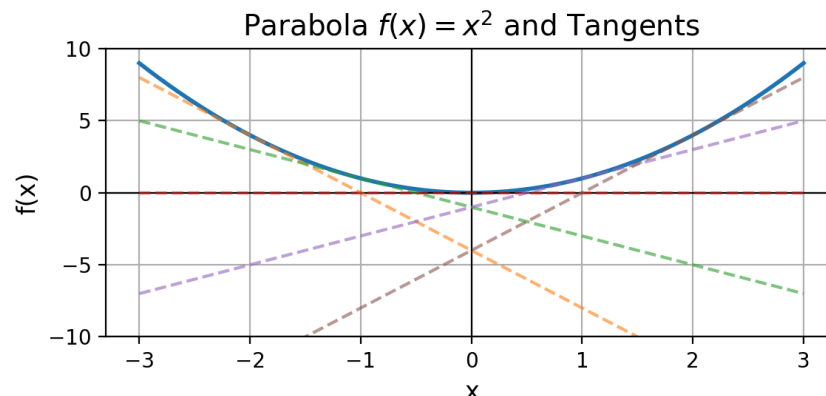
When the tangent line is horizontal, one of the following is true:

- The rate of change is 0: We are at a local *maximum*
- The rate of change is 0: We are at a local *minimum*
- The derivative is undefined

Symbolically:

$$f'(x) = 0 \quad \text{or} \quad f'(x) \text{ is undefined}$$

These points are called critical points. To determine the type of a critical point (maximum, minimum, or neither), we use the second derivative test.



The Second Derivative Test

The second derivative of a function tells us how the slope is changing near critical points.

If f is twice differentiable and x is a point where $f'(x) = 0$, then:

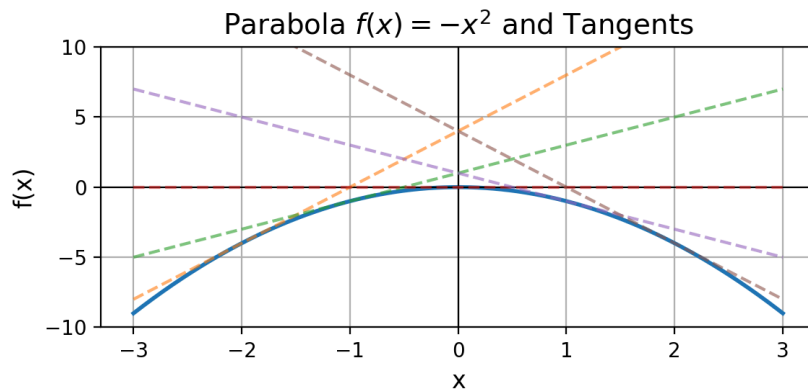
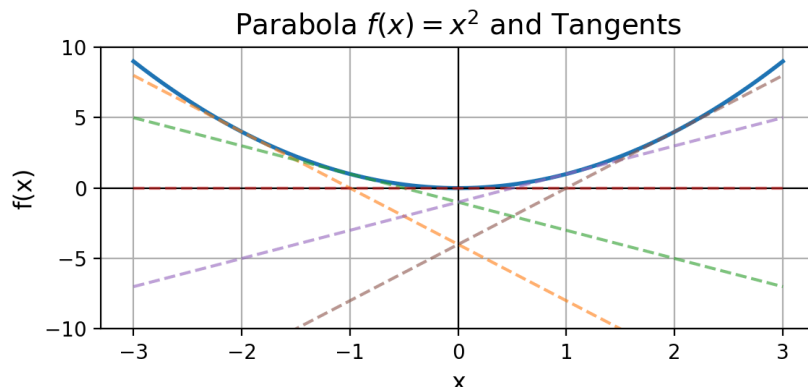
- If $f''(x) > 0$: local minimum
- If $f''(x) < 0$: local maximum
- If $f''(x) = 0$ or $f''(x)$ does not exist

Examples:

The point $x = 0$ is a critical point of:

$$f_1(x) = x^2 \rightsquigarrow f_1'(x) = 2x \rightsquigarrow f_1''(x) = 2$$

$$f_2(x) = -x^2 \rightsquigarrow f_2'(x) = -2x \rightsquigarrow f_2''(x) = -2$$



Exercise Set

- Part 1

Find the derivative of the following functions:

1. $h(x) = (x^2 + x)^{\frac{3}{2}}$

2. $h(x) = \frac{1}{(3x^2+1)^2}$

Find the local maxima or minima of the functions:

3. $f(x) = 2x^2 + 2x - 3$

4. $g(x) = 3x^3 + \frac{1}{4}x^2 - \frac{3}{2}x$

Use [desmos.com](https://www.desmos.com) to verify the result of problem 3 and 4.