Brush-Up Maths for Data Science (2025)

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Limits

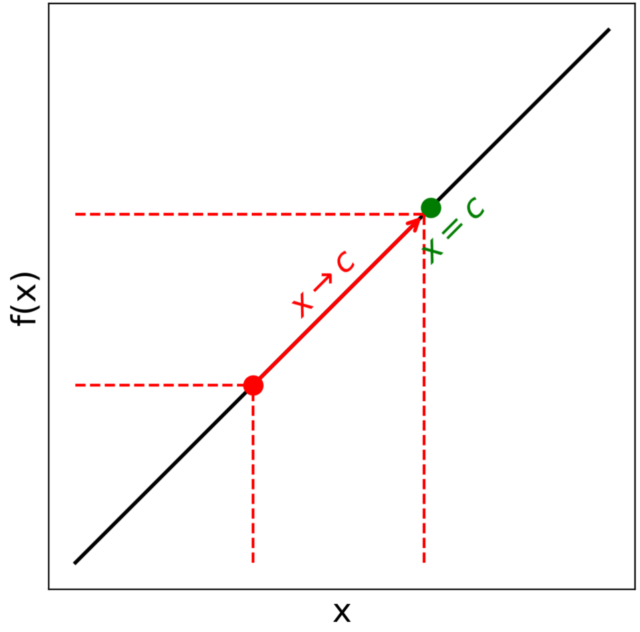
The concept of a limit concerns the value that a function $f(x)$ approaches as $x \to c$.

Note that:

- What happens exactly at c is not what matters
- What happens around the point c is what does

Why understand limits?

- Fundamental to study the continuity of functions
- Limits are the foundation for defining derivatives



Limits

- Building Intuition

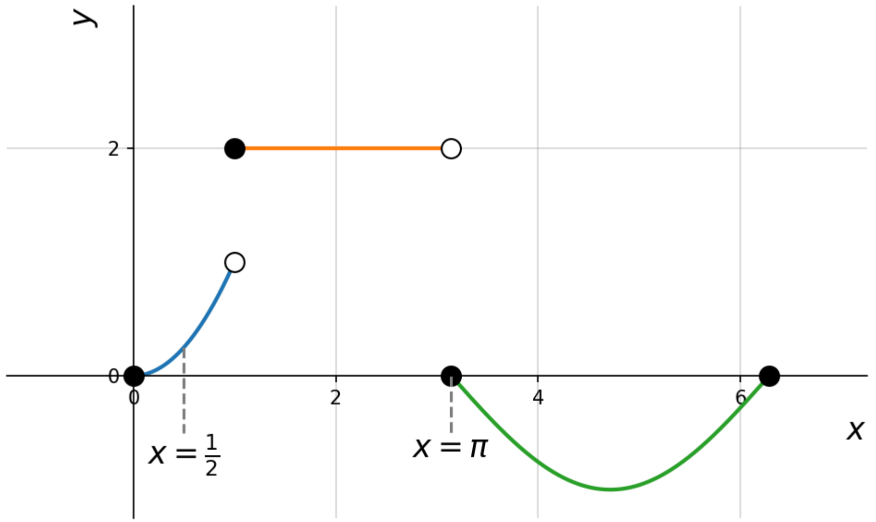
To illustrate and build an intuitive understanding, consider the function and its corresponding graph:

$$f(x) = \begin{cases} x^2, & x \in [0,1) \\ 2, & x \in [1,\pi) \\ \sin(x), & x \in [\pi, 2\pi] \end{cases}$$

Let us consider the limit of $f(x)$ as $x \to \frac{1}{2}$:

- Look at values of $f(x)$ for x around $\frac{1}{2}$.
- Examine the function x^2 on the interval $[0,1)$.

As x approaches $\frac{1}{2}$ from either the left or the right, the value of $f(x)$ approaches $\frac{1}{4}$.



The limit of $f(x)$ as $x \to \frac{1}{2}$ can be expressed as:

$$\lim_{x \to \frac{1}{2}} f(x) = \frac{1}{4}$$

Limits

- Definition

To more formally define the limit of a function, we let:

- f be a function on a domain $D \subseteq \mathbb{R}$
- c be a real number (not necessarily in D)
- L be a real number

In this case:

- As $x \in D$ gets closer to c (but not equal to x)
- If $f(x)$ gets closer and stays close to L

$\leadsto$ We say: The limit of $f(x)$ as x approaches c is L

Symbolically, we express this idea as:

$$\lim_{x \to c} f(x) = L$$

if and only if, it is the case that:

$$\lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L$$

Note:

- $x \to c^+$: Approaching from the positive direction
- $x \to c^-$: Approaching from the negative direction

Limits

- When They Do Not Exist

Note that:

- Not every function has a limit at every point
- A limit may fail to exist for several reasons

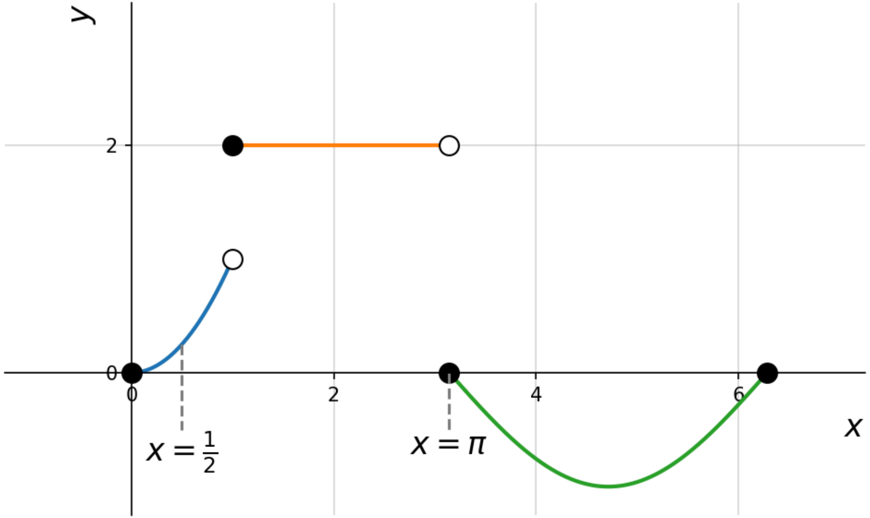
To illustrate this, consider:

- The earlier function and its graph
- The point $x = \pi$

The left-hand and right-hand limits differ:

$$L^- = \lim_{x \to \pi^+} f(x) = 2 \quad \text{and} \quad L^+ = \lim_{x \to \pi^-} f(x) = 0$$

$\rightsquigarrow \lim_{x \to \pi} f(x)$ does not exist. The same reasoning also applies to the point $x = 1$



Another way a limit can fail to exist is if the function grows without bound. For example: $\lim_{x \to \infty} x^2$

As $x \to \infty$, then $x^2 \to \infty$, i.e., the function does not approach a finite value.

Discontinuities

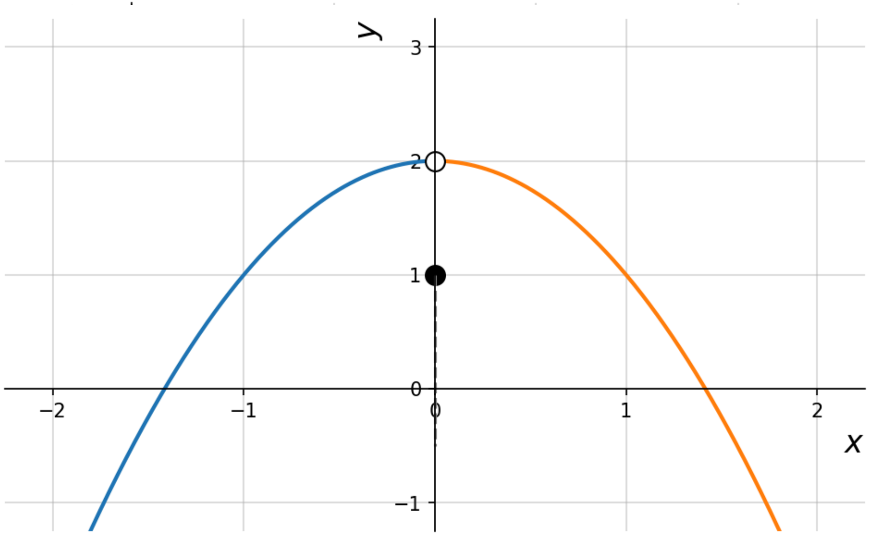
A discontinuity is a point where the function's value changes abruptly.

In our example, f has discontinuities at $x = 1$ and $x = \pi$ because the limits at these points do not exist.

However, a discontinuity does not always mean the limit fails to exist.

For instance, consider the function and its graph:

$$f(x) = \begin{cases} -x^2 + 2, & x \in \mathbb{R} \setminus \{0\} \\ 1, & x = 0 \end{cases}$$



It is clear that despite $\lim_{x \to 0} f(x) = 2$, the exact function value at $x = 0$ is $f(0) = 1$

$\rightsquigarrow$ We say that f is discontinuous at $x = 0$.

Continuity

- Definition

To define the continuity of a function, let:

- $f : D \to \mathbb{R}$ be a function
- $D \subseteq \mathbb{R}$ the domain of f

In this case, f is said to be continous at $c \in D$, if:

$$\lim_{x \to c} f(x) = f(c)$$

Breaking it down:

1. The limit exists:
$$\lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L$$
2. The function value $f(c)$ is defined
3. The limit equals the function value: $L = f(c)$

If f is continuous at every $c \in D$, then f is said to be continuous on D.

Note here, that:

- Continuity is a local property at each point
- Most common functions we care about are continuous on their domains

Continuity

- Example

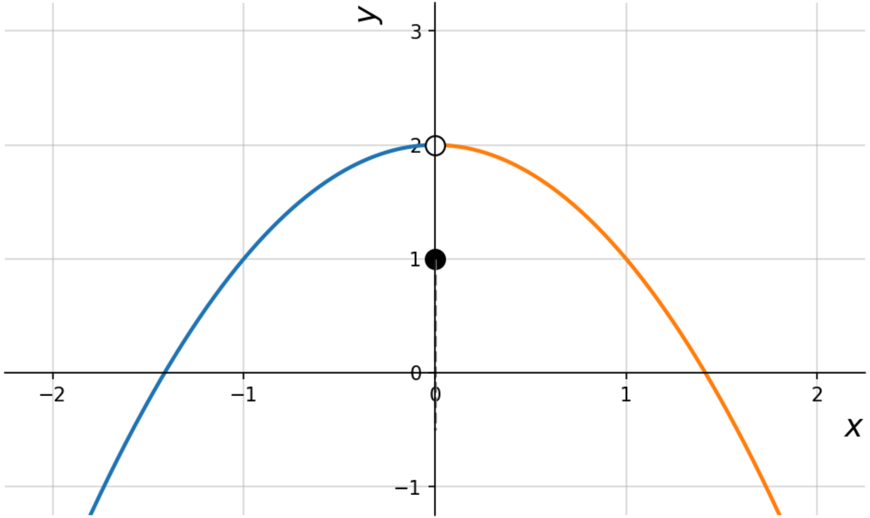
Recall the earlier function and its graph:

$$f(x) = \begin{cases} -x^2 + 2, & x \in \mathbb{R} \setminus \{0\} \\ 1, & x = 0 \end{cases}$$

Is the function continuous at $x = 0$?

We can break down the continuity condition:

1. The limit exists:
$$\lim_{x \to 0^+} f(x) = 2 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = 2 \quad \checkmark$$
2. The function value $f(0) = 1$ is defined ✓
3. The limit does not equal the function value:
$$\lim_{x \to 0} f(x) = 2 \neq f(0) = 1 \quad \times$$



Limit Laws

In an earlier chapter, we saw how to combine two functions using arithmetic operations. The table below shows the corresponding rules for limits, assuming both $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ exist.

Operation	Limit Law
Constant Multiple	$\lim_{x \to c}[a \cdot f(x)] = a \cdot \lim_{x \to c} f(x)$
Sum	$\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
Difference	$\lim_{x \to c}[f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
Product	$\lim_{x \to c}[f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
Quotient	$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}, \quad \lim_{x \to c} g(x) \neq 0$

Limit Laws

- Examples

Determine $\lim_{x\to 1}\left(\sqrt{x+3}-\sqrt{4-x}\right)$

1. Split into two limits (use the difference law):
$$\lim_{x\to 1}\sqrt{x+3}-\lim_{x\to 1}\sqrt{4-x}$$
2. Evaluate each part directly (direct substitution):
$$\begin{aligned}\lim_{x\to 1}\sqrt{x+3}-\lim_{x\to 1}\sqrt{4-x} &= \sqrt{1+3}-\sqrt{4-1}\\ &= \sqrt{4}-\sqrt{3}\\ &= 2-\sqrt{3}\end{aligned}$$

Determine $\lim_{x\to 1}\frac{x^2-1}{x-1}$

1. Check direct substitution (does not work!):
$$\frac{1^2-1}{1-1}=\frac{0}{0}$$
2. Factor numerator and cancel the common factor:
$$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}\\ &= \lim_{x\to 1}x+1\end{aligned}$$

4. Evaluate the expression (direct subsitution):
$$\begin{aligned}\lim_{x\to 1}x+1 &= 1+1\\ &= 2\end{aligned}$$

Exercise Set

- Part 1

Determine the limit (if it exists):

1. $\lim_{x \to 2} 2x$
2. $\lim_{x \to \infty} x$
3. $\lim_{x \to 1} x^2 + x$
4. $\lim_{x \to 2} 2x^3 - 2e^x$
5. $\lim_{x \to \infty} \frac{1}{x}$ (argue intuitively, e.g. based on a graph)
6. $\lim_{x \to 0} \frac{1}{x}$ (argue intuitively, e.g. based on a graph)

Consider the function $f : \mathbb{R}_{>0} \to \mathbb{R}^+$ defined by:

$$f(x) = \frac{1}{x}$$

7. Is the function continuous?
8. What if the domain is $\mathbb{R}$ instead of $\mathbb{R}_{>0}$?

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ -x + 4 & \text{if } x > 1 \end{cases}$$

9. Determine whether it is continuous at $x = 1$. If it is not, then what condition does it fail to comply with?