Brush-Up Maths for Data Science (2025)

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Equation Solving

Functions have particular points of interes on their graphs due to what they represent:

- The x-intercepts are the points at which the output value is zero
- The y-intercept is the point at which the function has an input value of zero

Analytically, these points can be found by solving:

- x-intercepts: solve $f(x) = 0$
- y-intercept: solve $f(0) = y$

Both of these tasks are examples of equation solving.

To solve an equaltion, we "isolate" a particular variable on one side of the equality:

$$x = "\text{stuff}"$$

where $"\text{stuff}"$ can be an expression containing numbers, constants, other variables, mathematical operators, etc.

Solutions to Equations as Roots

A root of a function $f(x)$ is synonymous with:

- An x-intercept of the graph of $f(x)$
- A value of x such that $f(x) = 0$

We can use this idea, noting that any equation of the form:

$$f(x) = g(x)$$

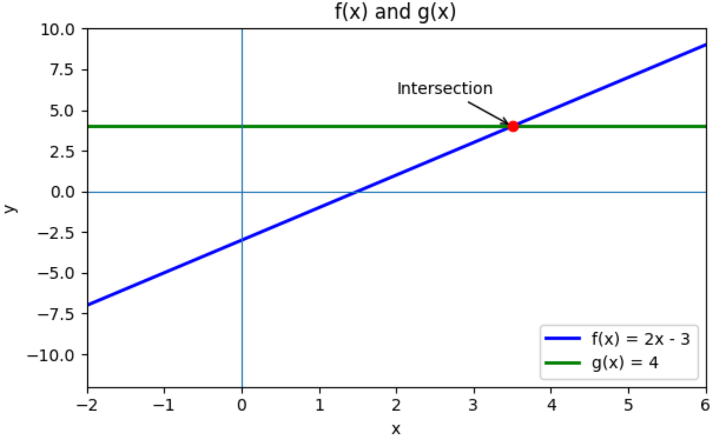
Can be rewritten by defining a new function:

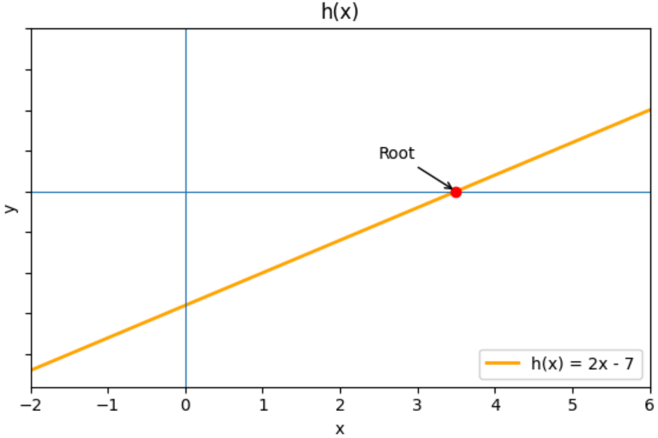
$$h(x) = f(x) - g(x).$$

The key idea:

Solving for the equality of two functions is equivalent to finding the x-intercepts of their difference.

Solving Linear Equations





Consider functions:

- $f(x) = 2x - 3$ (linear)
- $g(x) = 4$ (constant)

Finding their intersection means determining x s.t:

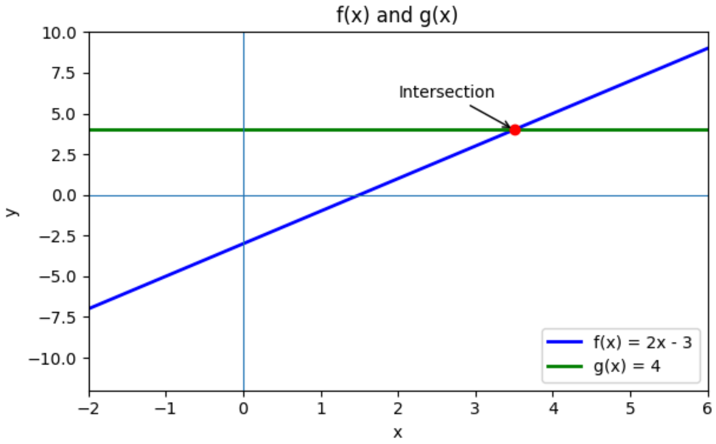
$$f(x) = g(x) \quad \Leftrightarrow \quad 2x - 3 = 4$$

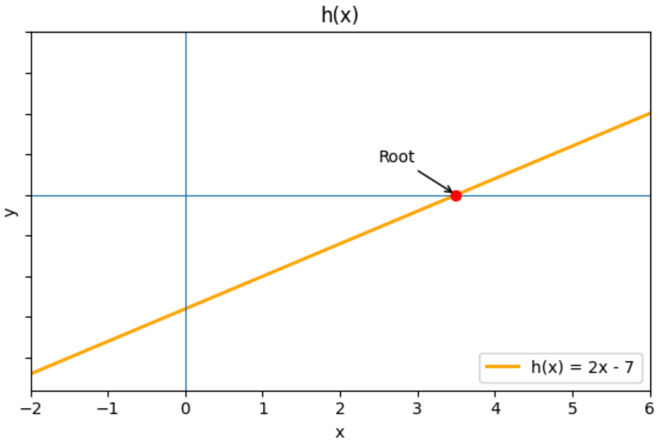
We can convert this to a root-finding problem by moving everything to the left-hand-side (LHS):

$$\begin{aligned} & f(x) - g(x) = 0 \\ \Leftrightarrow \quad & 2x - 3 - 4 = 0 \\ \Leftrightarrow \quad & 2x - 7 = 0 \end{aligned}$$

The LHS can be regarded as a new function $h(x)$.

Solving Linear Equations





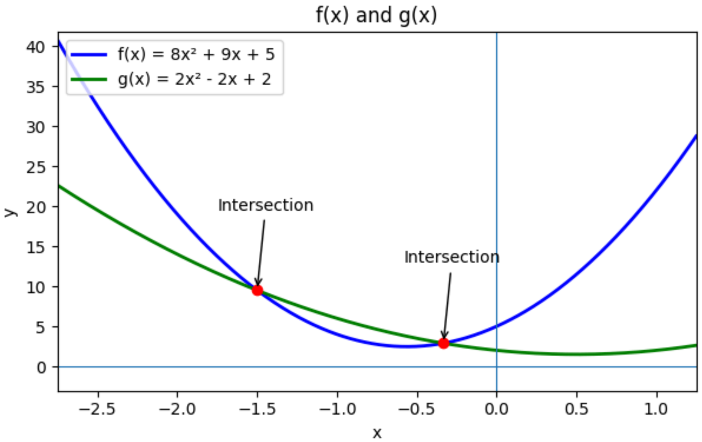
Finding its root is the same as solving the original equation:

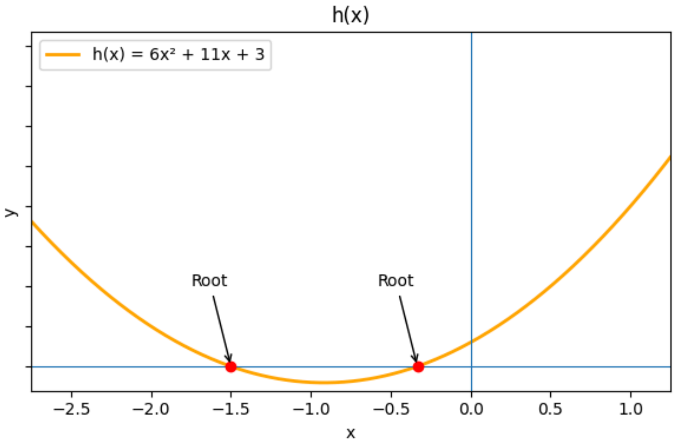
$$
\begin{aligned}
& h(x) = 2x - 7 = 0 \\
\Leftrightarrow \quad & 2x = 7 \\
\Leftrightarrow \quad & \frac{2x}{2} = \frac{7}{2} \\
\Leftrightarrow \quad & x = 3.5
\end{aligned}
$$

$\rightsquigarrow$ The algebraic solution matches the geometric interpretation:

- We found that the solution is $x = 3.5$
- The graphs intersect at the same point

Solving Quadratic Equations





Consider functions:

- $f(x) = 8x^2 + 9x + 5$ (quadratic)
- $g(x) = 2x^2 - 2x + 2$ (quadratic)

Finding their intersection means determining x s.t:

$$f(x) = g(x) \quad \Leftrightarrow \quad 8x^2 + 9x + 5 = 2x^2 - 2x + 2$$

We can convert this to a root-finding problem by moving everything to the left-hand-side (LHS):

$$\begin{aligned} f(x) - g(x) &= 0 \\ \Leftrightarrow \quad (8x^2 + 9x + 5) - (2x^2 - 2x + 2) &= 0 \\ \Leftrightarrow \quad 8x^2 + 9x + 5 - 2x^2 + 2x - 2 &= 0 \\ \Leftrightarrow \quad 6x^2 + 11x + 3 &= 0 \end{aligned}$$

The LHS can be regarded as a new function $h(x)$.

Solving Quadratic Equations

- Via Factorization

We factor the trinomial:

$$h(x) = 6x^2 + 11x + 3$$

Using factorization by grouping method:

1. Identify coefficients: $a = 6, b = 11, c = 3$.
2. Find integers m, n such that $a \cdot c = 6 \cdot 3 = 18$.
   - Choose $m = 9$ and $n = 2$, since:
   - $m \cdot n = 18$
   - $m + n = 11$
3. Rewrite the middle term:

$$6x^2 + 11x + 3 = 6x^2 + \underbrace{9x}_{mx} + \underbrace{2x}_{nx} + 3$$

We can now proceed by grouping.

4. Group into pairs:

$$(6x^2 + 9x) + (2x + 3)$$

5. Factor the GCF from each group:

$$3x(2x + 3) + 1(2x + 3)$$

6. A Common binomial factor appears:

$$(2x + 3)(3x + 1)$$

7. We can now solve $h(x) = 0$, by solving each factor:

$$2x + 3 = 0 \ \Rightarrow \ x_1 = -\frac{3}{2}$$

$$3x + 1 = 0 \ \Rightarrow \ x_2 = -\frac{1}{3}$$

Solving Quadratic Equations

- The Quadratic Formula

Consider the quadratic equation:

$$ax^2 + bx + c = 0, \quad a \neq 0$$

The solutions of this equation are given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $\Delta = b^2 - 4ac$ determines the number of real solutions:

- If $\Delta > 0$: two distinct real solutions.
- If $\Delta = 0$: one real (repeated) solution.
- If $\Delta < 0$: no real solutions.

Note: The $\pm$ symbol in the formula above indicates that we consider both the negative and positive square root.

Solving Quadratic Equations

- Via The Discriminant

To solve the quardratic equation

$$h(x) = 0 \quad \Leftrightarrow \quad 6x^2 + 11x + 3 = 0$$

We set a = 6, b = 11, and c = 3 in the formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-11 \pm \sqrt{11^2 - 4 \cdot 6 \cdot 3}}{2 \cdot 6} \\ &= \frac{-11 \pm \sqrt{121 - 72}}{12} \\ &= \frac{-11 \pm \sqrt{49}}{12} \end{aligned}$$

Hence, we get:

$$\begin{aligned} x_1 &= \frac{-11 + 7}{12} \\ &= -\frac{4}{12} = -\frac{1}{3} \end{aligned}$$

Or on the other hand:

$$\begin{aligned} x_2 &= \frac{-11 - 7}{12} \\ &= -\frac{18}{12} = -\frac{3}{2} \end{aligned}$$

These match the solutions obtained by factoring.

Verifying Solutions

- Example

Whenever:

- We have solved an equation to obtain a solution
- We can always substitute the solution back into the equation to verify it

For example:

Consider the equation from earlier:

$$h(x) = 6x^2 + 11x + 3 = 0$$

With solution:

$$x_1 = -\frac{3}{2} \quad \text{and} \quad x_2 = -\frac{1}{3}$$

Substituting the solution x_1 into the equation we see:

$$
\begin{aligned}
6 \cdot \left(-\frac{3}{2}\right)^2 + 11 \cdot \left(-\frac{3}{2}\right) + 3 &= 0 \\
6 \cdot \left(\frac{9}{4}\right) + 11 \cdot \left(-\frac{3}{2}\right) + 3 &= 0 \\
\frac{54}{4} + \left(-\frac{33}{2}\right) + 3 &= 0 \\
\frac{27}{2} - \frac{33}{2} + 3 &= 0 \\
-\frac{6}{2} + 3 &= 0 \\
-3 + 3 &= 0 \quad \checkmark
\end{aligned}
$$

Verifying Solutions

- Example (Continued)

Whenever:

- We have solved an equation to obtain a solution
- We can always substitute the solution back into the equation to verify it

For example:

Consider the equation from earlier:

$$h(x) = 6x^2 + 11x + 3 = 0$$

With solution:

$$x_1 = -\frac{3}{2} \quad \text{and} \quad x_2 = -\frac{1}{3}$$

Substituting the solution x_2 into the equation we see:

$$6 \cdot \left(-\frac{1}{3}\right)^2 + 11 \cdot \left(-\frac{1}{3}\right) + 3 = 0$$

$$6 \cdot \left(\frac{1}{9}\right) + 11 \cdot \left(-\frac{1}{3}\right) + 3 = 0$$

$$\frac{6}{9} + \left(-\frac{11}{3}\right) + 3 = 0$$

$$\frac{2}{3} - \frac{11}{3} + 3 = 0$$

$$-\frac{9}{3} + 3 = 0$$

$$-3 + 3 = 0 \quad \checkmark$$

Inverse Functions

- Definition

When solving an equation of the form:

$$f(x) = k$$

We want a general way to determine x for any value k.

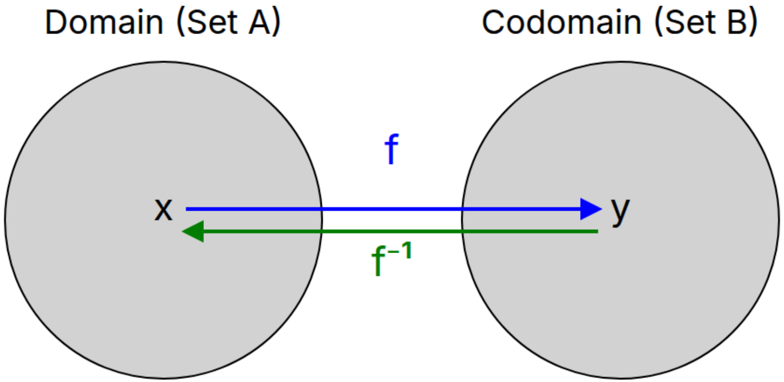
For certain functions, it is possible to a function that "reverses" this mapping.

Let $f : A \to B$ be a function. An inverse function $f^{-1} : B \to A$ then satisfies:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$$

for every $x \in A$ and B.

Note: that an inverse function exists only if f is bijective (both one-to-one and onto).



Inverse Functions

- Definition (Continued)

The inverse function essentially allows us to solve equations by applying f^{-1} to both sides:

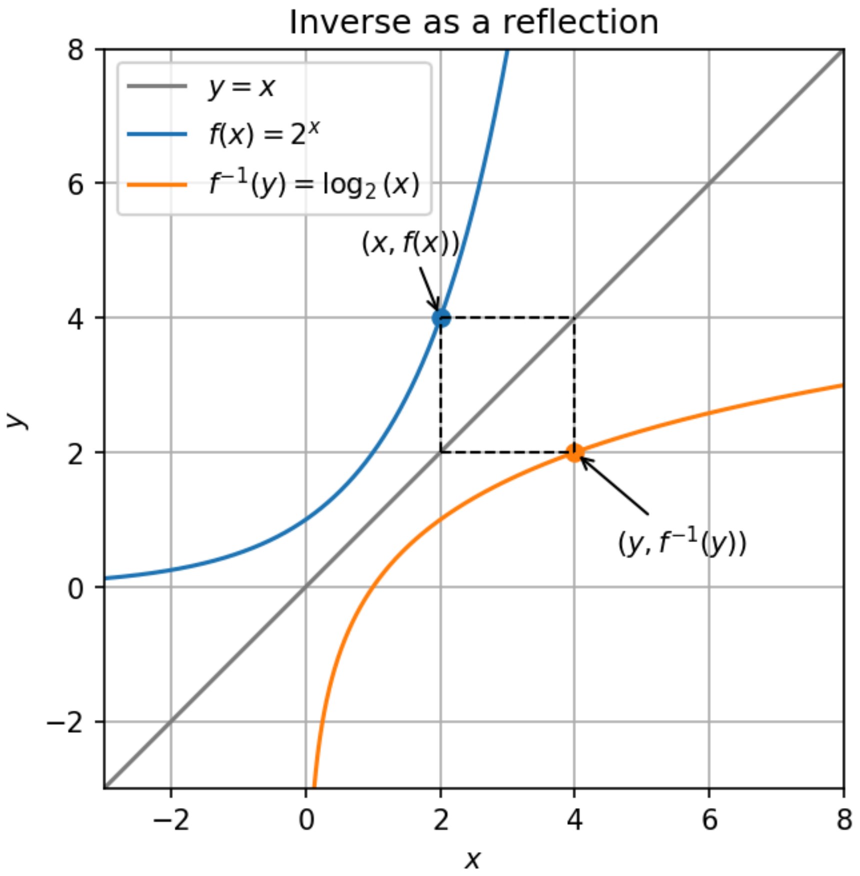
$$f(x) = y \quad \Rightarrow \quad x = f^{-1}(y)$$

The composition of a function and its inverse returns the identity function on the respective domains:

$$f \circ f^{-1} = \mathrm{id}_B, \quad f^{-1} \circ f = \mathrm{id}_A.$$

$\rightsquigarrow$ The inverse of a function f reverses the domain and codomain of f.

Graphically, the inverse corresponds to reflecting the graph of f across the line $y = x$.



Inverse Functions

- Example

Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = 3x + 5.$$

To find f^{-1}, solve for x in terms of y:

$$y = 3x + 5 \quad \Rightarrow \quad x = \frac{y-5}{3}.$$

This expression tells us how to recover x from a given output y, so:

$$f^{-1}(y) = \frac{y-5}{3}.$$

Now suppose we want to solve

$$f(x) = 14$$

To isolate x, we apply f^{-1} to both sides:

$$f^{-1}(f(x)) = f^{-1}(14)$$

Since f^{-1} reverses the action of f, the left-hand side simplifies to x:

$$x = \frac{14-5}{3} = 3$$

This is the reason for applying f^{-1} to both sides: it “undoes” f on the side containing x, leaving x alone.

Common Inverse Functions

Function $f(x)$	Inverse $f^{-1}(x)$	Domain of f	Range of f
$f(x) = x + a$	$f^{-1}(x) = x - a$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = x - a$	$f^{-1}(x) = x + a$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = kx$, $k \neq 0$	$f^{-1}(x) = \frac{x}{k}$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = \frac{x}{k}$, $k \neq 0$	$f^{-1}(x) = kx$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = x^n$, n odd	$f^{-1}(x) = \sqrt[n]{x}$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = x^n$, n even	$f^{-1}(x) = \sqrt[n]{x}$ (principal root)	$[0, \infty)$	$[0, \infty)$
$f(x) = e^x$	$f^{-1}(x) = \ln(x)$	$\mathbb{R}$	$(0, \infty)$
$f(x) = a^x$, $a > 0, a \neq 1$	$f^{-1}(x) = \log_a(x)$	$\mathbb{R}$	$(0, \infty)$
$f(x) = \ln(x)$	$f^{-1}(x) = e^x$	$(0, \infty)$	$\mathbb{R}$
$f(x) = \log_a(x)$, $a > 0, a \neq 1$	$f^{-1}(x) = a^x$	$(0, \infty)$	$\mathbb{R}$

Note: Functions like the trigonometric, and hyberbolic functions are also common to encounter, but are left due to conciseness.

Solving Non-Linear Equations

What are non-linear equations?

- Equations involving polynomials
- Trigonometric and hyperbolic functions
- Exponentials and logarithms
- etc…

How do we solve them?

- Transform the equation to isolate the variable
- Check domain restrictions to validate the solution

Key strategy:

- Undo the operation affecting the variable

We will focus on non-linear equations involing:

- Exponentials
- Logarithms

With these equations, the typical solution procedure is:

- If the variable is in the exponent:
  - $\rightsquigarrow$ apply a logarithm to both sides
- If the variable is inside a logarithm:
  - $\rightsquigarrow$ apply an exponential to both sides

Logarithm Rules

- Logarithm Rules

For $a > 0, b > 0, n \in \mathbb{R}$, and $a, b \neq 1$, the most important rules are given in the following table.

Rule	Formula	Description
Product Rule	$\log_b(xy) = \log_b(x) + \log_b(y)$	The logarithm of a product equals the sum of the logarithms.
Quotient Rule	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$	The logarithm of a quotient equals the difference of the logarithms.
Power Rule	$\log_b(x^n) = n\log_b(x)$	A power in the argument becomes a multiplier in front of the logarithm.
Logarithm of 1	$\log_b(1) = 0$	Any base raised to the power 0 equals 1.
Logarithm of the Base	$\log_b(b) = 1$	Any base raised to the power 1 equals itself.
Inverse Property	$b^{\log_b(x)} = x$	Exponential and logarithmic functions cancel each other.
Natural Log of e	$\ln(e) = 1$	Since $\ln(x)$ means $\log_e(x)$.
Change of Base	$\log_b(x) = \frac{\log_k(x)}{\log_k(b)}$	Converts a logarithm from one base to another.

Note: The natural logarithm $\ln(x)$ is simply $\log_e(x)$, where $e \approx 2.71828$. All these rules work the same way for $\ln$ as for $\log_b$ for any base $b > 0, b \neq 1$.

Solving Non-Linear Equations

- Examples

Let us solve the following equation for x:

$$e^{2x+1} = w, \quad w > 0$$

We proceed as follows:

$$\begin{aligned} e^{2x+1} &= w \\ \Leftrightarrow \ln\left(e^{2x+1}\right) &= \ln(w) \\ \Leftrightarrow 2x+1 &= \ln(w) \\ \Leftrightarrow x &= \frac{1}{2}\left(\ln(w) - 1\right) \end{aligned}$$

Let us solve the equation following equation for x:

$$\ln(x^2 - 10) = 6$$

Assuming that $x^2 > 10$, we obtain:

$$\begin{aligned} \ln(x^2 - 10) &= 6 \\ \Leftrightarrow e^{\ln(x^2-10)} &= e^6 \\ \Leftrightarrow x^2 - 10 &= e^6 \\ \Leftrightarrow x^2 &= e^6 + 10 \\ \Leftrightarrow x &= \pm\sqrt{e^6 + 10} \end{aligned}$$

The $\pm$ symbol indicates that both the positive and negative number is a solution.

This is the case since $1^2 = 1$ and $(-1)^2 = 1$.

Determining Whether a Relation is a Function

- Graphically

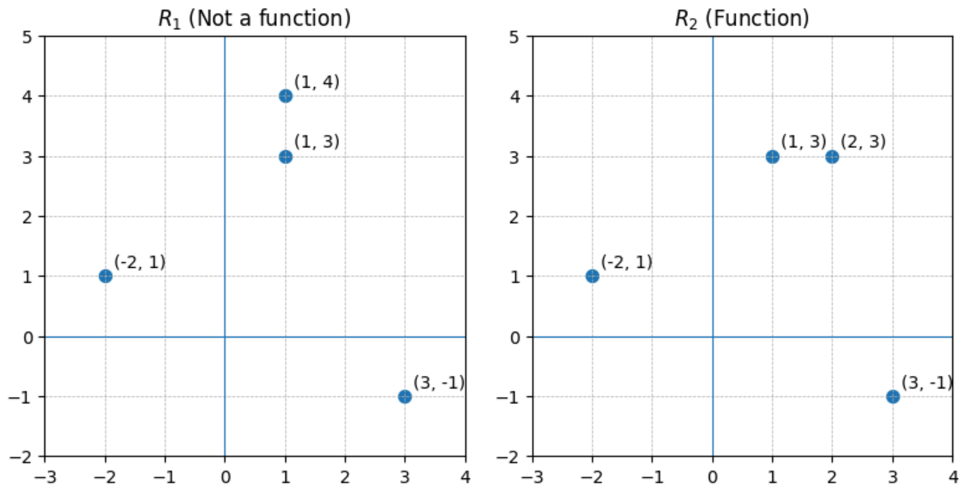
A relation in which each x-coordinate is matched with exactly one y-coordinate is said to describe y as a function of x.

This also means that, if the same x-coordinate is associated with two different y-coordinates, then the relation is not a function.

Example:

Which of the following relations descbribe y as a function of x?

- $R_1 = \{(-2, 1), (1, 3), (1, 4), (3, -1)\}$
- $R_2 = \{(-2, 1), (1, 3), (2, 3), (3, -1)\}$



Determining Whether a Relation is a Function

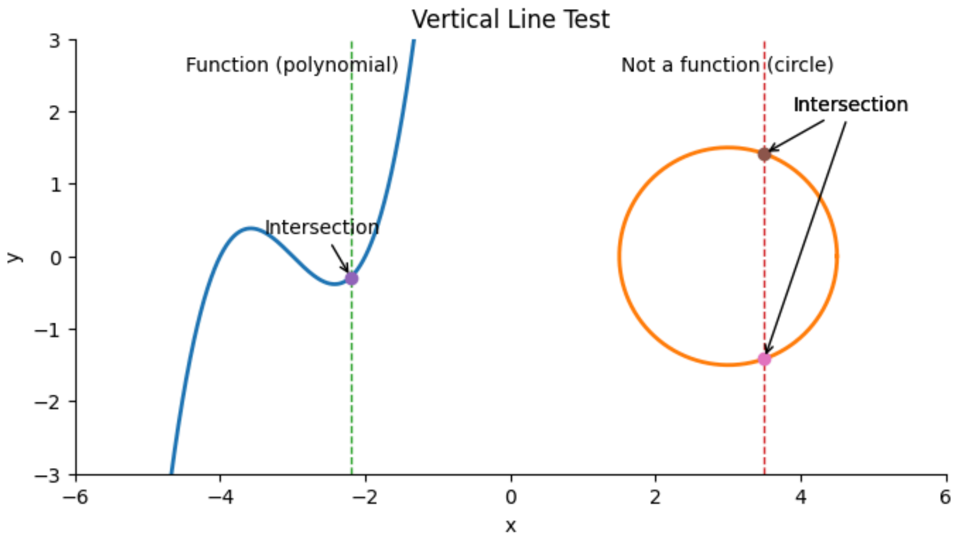
- The Vertical Line Test

This leads to the vertical line test, which is a graphical method to decide whether a relation is a function:

- A relation is a function if and only if every vertical line intersects its graph at most once.
- If a vertical line intersects more than once, the relation assigns more than one output to the same input thus violating the definition of a function.

Note: Equations can describe valid relationships, like the shape of a circle, but do not define a function.

Recognizing this helps us understand both the limits of function notation and the situations where we need use other representations.



Determining Whether a Relation is a Function

- Algebraically

We can check whether an equation defines a function by solving for one variable in terms of the other.

If solving produces more than one output value for the same input, then the relation is not a function.

Example:

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution:

First we subtract x^2 from both sides:

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation:

$$y = \pm\sqrt{1 - x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$.

Exercise Set

- Part 1

Solve each of the equations for x and verify the solution:

1. $4 = 2x$
2. $2x + 5x - 8 = 20$
3. $5^{3x-2} = 5$
4. $\log_5 (2x + 3) = 2$

Find the roots of the polynomial $f(x) = 2x^2 + 5x + 2$ using:

5. Factorization
6. The discriminant

Determine, by solving, which equations represent y as a function of x:

7. $x^3 + y^2 = 1$
8. $x^2 + y^3 = 1$

Use desmos.com to verify the result of problem 7 and 8.