Brush-Up Maths for Data Science (2025)

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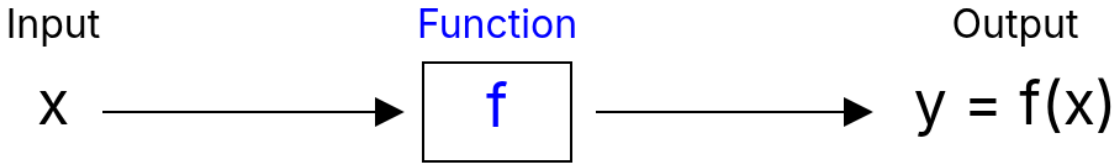
Functions

- Intuition & Terminology



A function can be thought of as an input/output device (f):

- Input: x
- Output: $y = f(x)$ is uniquely determined

We often call the:

- Input: x the independent variable
- Output: y the dependent variable

Functions

- Definition

A function f is a relation between two sets:

- A (the domain of f)
- B (the codomain of f)

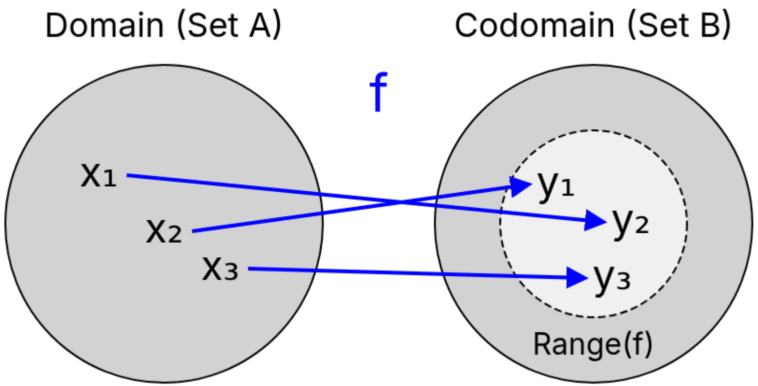
Formally we write:

$$f : A \to B$$

In particular:

- A contains all valid inputs.
- B is the set into which all outputs are mapped.
- $\text{Range}(f)$ is the set of outputs the function produces based on A. It is a subset of the codomain:

$$\text{Range}(f) \subseteq B$$



Functions

- Examples

Example 1: Area of a square

- $A(s) = s^2$, with $s =$ side length
- Domain: $s \in \mathbb{R}_{\geq 0} = [0, \infty)$
- Codomain: $\mathbb{R}$
- Range: $\mathbb{R}_{\geq 0} = [0, \infty)$

Example 2: Temperature over a day

- $T(t) =$ temperature at time t
- Domain: $t \in [0, 24]$
- Codomain: $\mathbb{R}$
- Range: A subset of $\mathbb{R}$ (e.g., $[10, 28]$)

Example 3: Distance traveled at constant speed

- $D(t) = 60 \cdot t$, with $t =$ time (hours)
- Domain: $t \in [0, \infty)$
- Codomain: $\mathbb{R}$
- Range: $[0, \infty)$

Example 4: Crop yield vs. fertilizer

- $f(x) =$ crop yield (in t/ha) for fertilizer amount x
- Domain: $x \in [0, \infty)$
- Codomain: $\mathbb{R}$
- Range: A subset of $\mathbb{R}$ (e.g., $[0, 5]$)

Functions

- Representation Methods

Functions can be represented in different ways:

- Each representation highlights different insights
- Choice depends on context and purpose

To illustrate these representations, we will use the simple example from earlier:

$$f(x) = \text{ crop yield for fertilizer amount } x$$

That is, $f(x)$ denotes the crop yield (t/ha) as a function of fertilizer amount x (kg/ha).

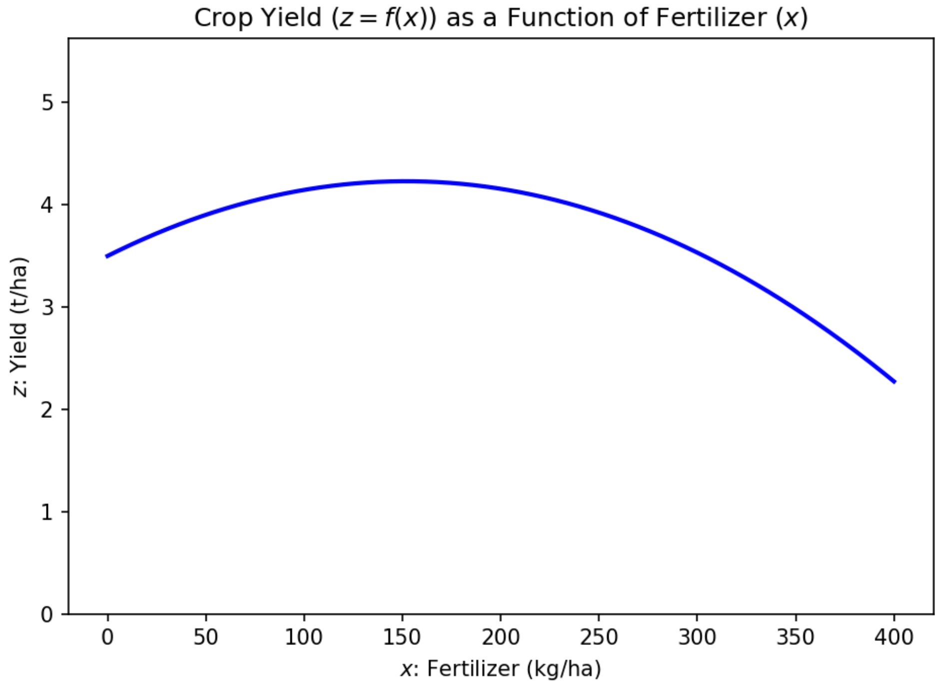
Representation Methods

- Tables

Fertilizer x (kg/ha)	Crop Yield $f(x)$ (t/ha)
0	3.4942
1	3.5038
$\cdots$	$\cdots$
197	4.1589
198	4.1559
$\cdots$	$\cdots$
396	2.3319
397	2.3163

Representation Methods

- Graphs



A picture of a function can be provided in the form of a graph where the set of points $(x, f(x))$ are plotted in a coordinate plane, for each x in the domain of f.

Representation Methods

- Algebraic Formulas

An algebraic formula provides:

- A general rule for any valid input
- Compact, symbolic way to describe relationships

The table and graph, shown earlier, have been generated based on the function:

$$f(x) = -3.17042 \cdot 10^{-5} x^2 + 0.00961968x + 3.49418$$

This formula is actually not arbitrary it has been fitted to observed data.

Benefits:

- Interpolation: Predict values between known data points
- Extrapolation: Determine the behavior of the function for very large or very small fertizlier amounts (x)
- Equation solving: Solve equations such as $f(x) = 4.0$ to find the corresponding fertilizer amount (x).

Basic Classes of Functions

Functions can be grouped into different classes based on their algebraic form.

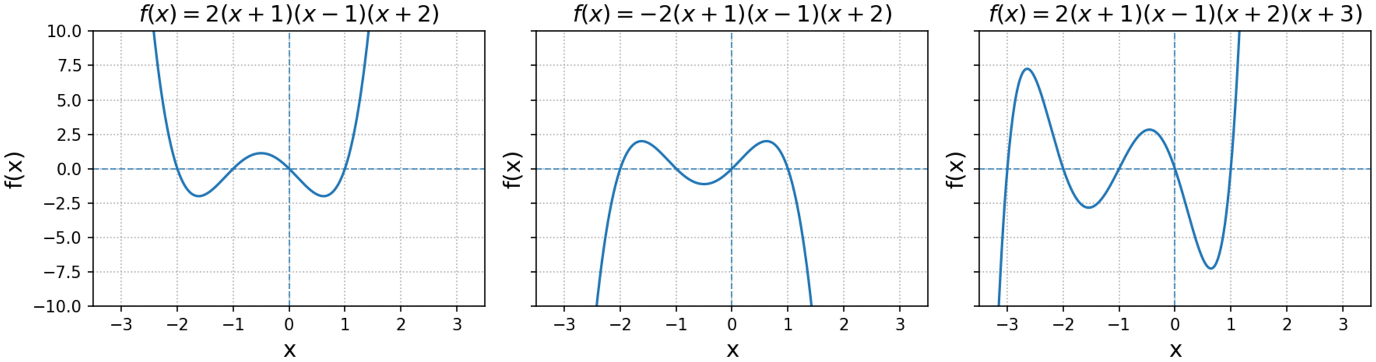
Each class has its own properties, domain and range, and characteristic graph shape.

Some common and important features are:

Feature	Definition
Intercepts	Points where the graph crosses the coordinate axes.
Turning Points	Points where the graph changes direction from increasing to decreasing, or vice versa.
Asymptotes	Lines that the graph approaches but does not cross : horizontal, vertical, or oblique.

Polynomial Functions

- Definition



Polynomials belong to a broad class of functions that can be written in the general form:

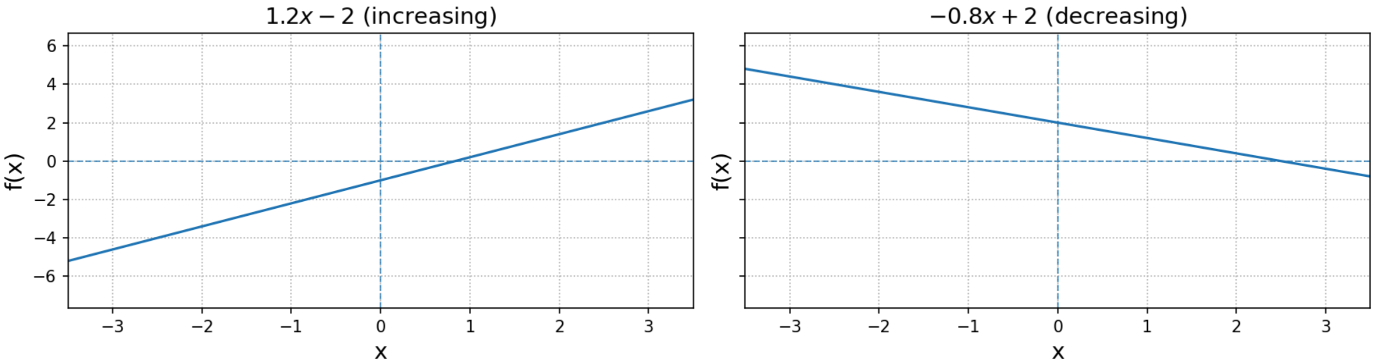
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Here:

- n is a non-negative integer (the degree of the polynomial)
- $a_n, a_{n-1}, \ldots, a_0$ are real constants
- $a_n \neq 0$ if $n > 0$

Polynomial Functions (Degree 1)

- Definition



A linear function is a polynomial of degree 1; its graph is a straight line and can be written in the general form:

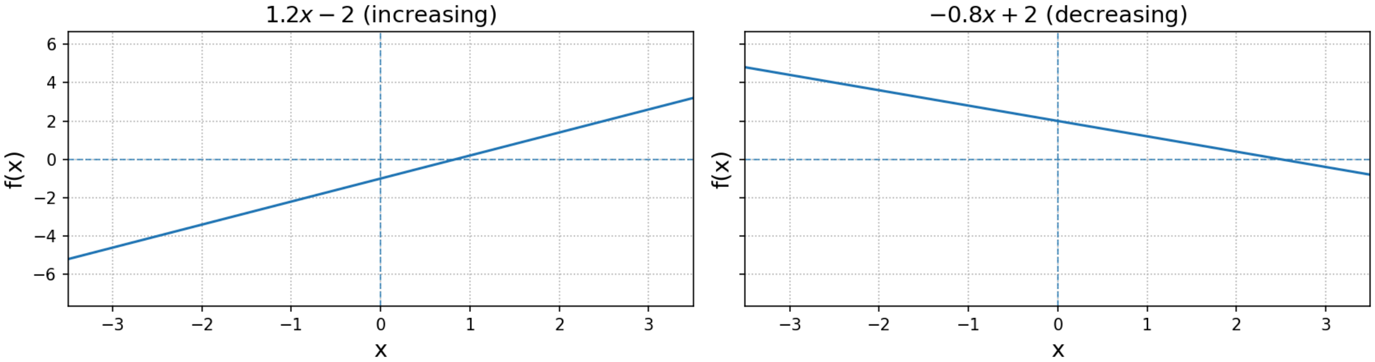
$$f(x) = ax + b$$

Here:

- a and b are real constants
- $a \neq 0$

Polynomial Functions (Degree 1)

- Definition



The general form of a linear function:

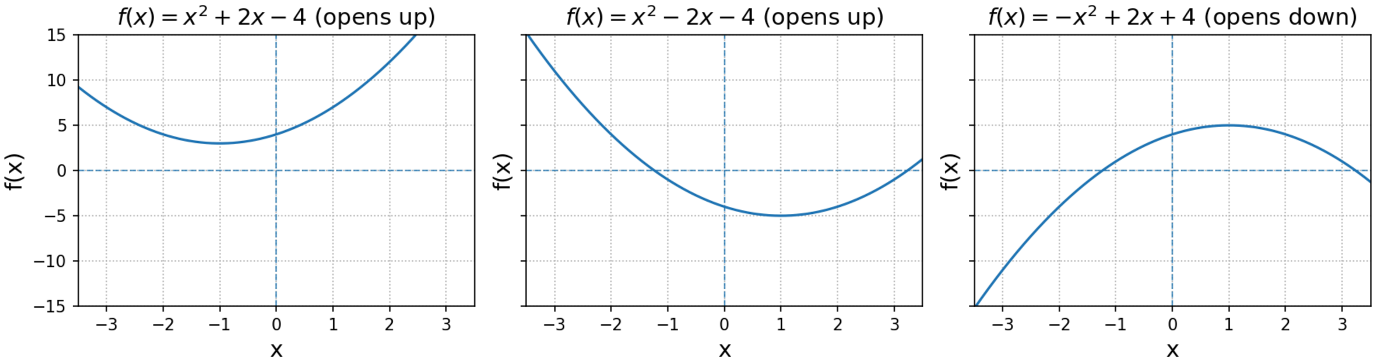
$$f(x) = ax + b$$

- Domain: All real numbers ($\mathbb{R}$)
- Range: All real numbers ($\mathbb{R}$)

- Graph: A straight line with slope a:
  - If $a > 0$ the function is increasing
  - If $a < 0$ the function is decreasing
- Intercepts:
  - y-intercept at $(0, b)$
  - x-intercept at $(\frac{-b}{a}, 0)$

Polynomial Functions (Degree 2)

- Definition



A quadratic function is a polynomial of degree 2; its graph is a parabola and can be written in the general form:

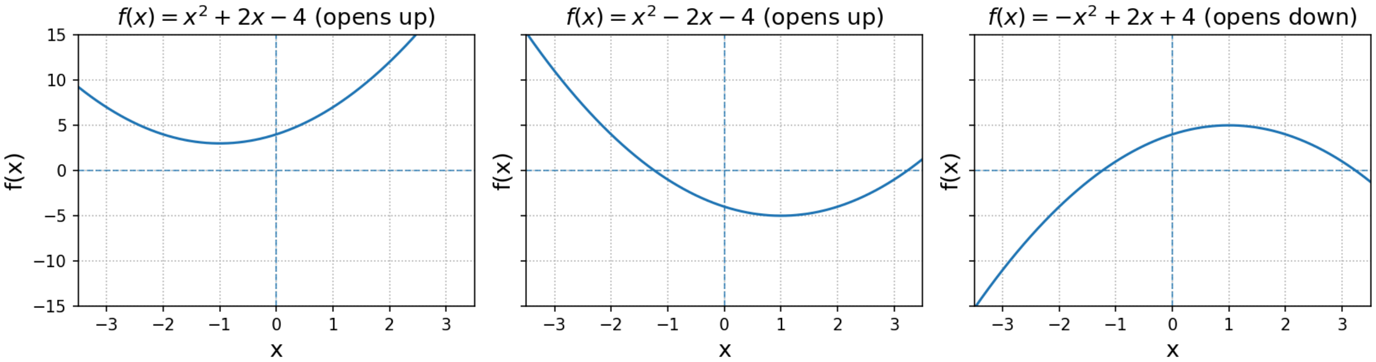
$$f(x) = ax^2 + bx + c$$

Here:

- a, b, and c are real constants
- $a \neq 0$

Polynomial Functions (Degree 2)

- Definition



The general form of a quadratic function:

$$f(x) = ax^2 + bx + c$$

- Domain: All real numbers ($\mathbb{R}$)
- Range:
  - If $a > 0$ it is $[y_{\min}, \infty)$
  - If $a < 0$ it is $(-\infty, y_{\max}]$

- Graph: Opening up ($a > 0$) or down ($a < 0$)
- Intercepts: Max 2 x-intercepts, exactly 1 y-intercept
- Turning Point:
  - If $a > 0$ it is the highest point $(x_{\max}, y_{\max})$
  - If $a < 0$ is is the lowest point $(x_{\min}, y_{\min})$

Polynomial Functions

- Terminology

Characterizing polynomialas based on **the degree**:

Degree	Name	Example
0	Constant	7
1	Linear	$2x + 3$
2	Quadratic	$x^2 - 4x + 4$
3	Cubic	$x^3 - x$
4	Quartic	$x^4 + 2x^2 + 1$
5	Quintic	$x^5 - x^3 + 1$
$n \geq 6$	nth-degree polynomial	$x^6 + \ldots$

Polynomial Functions

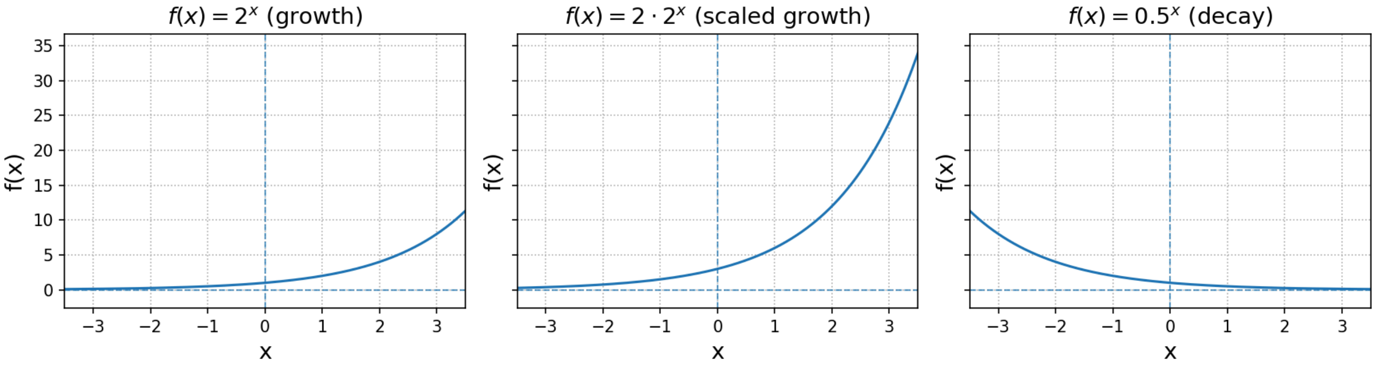
- Terminology (Continued)

Characterizing polynomialas based on **the number of terms**:

Number of Terms	Name	Example
1	Monomial	$5x^3$
2	Binomial	$3x^2 + 1$
3	Trinomial	$x^2 - 4x + 4$
≥ 4	Polynomial	$x^4 + x^3 - 2x + 7$

Exponential Functions

- Definition



Exponential functions have a constant base raised to a variable exponent and can be written in the general form:

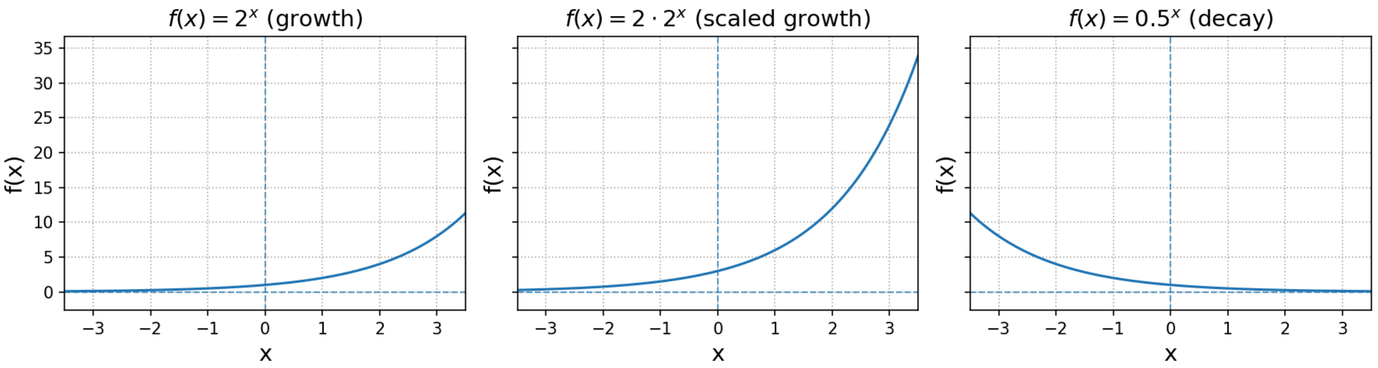
$$f(x) = ab^x$$

Here:

- $a \neq 0$
- $b > 0$, and $b \neq 1$

Exponential Functions

- Definition



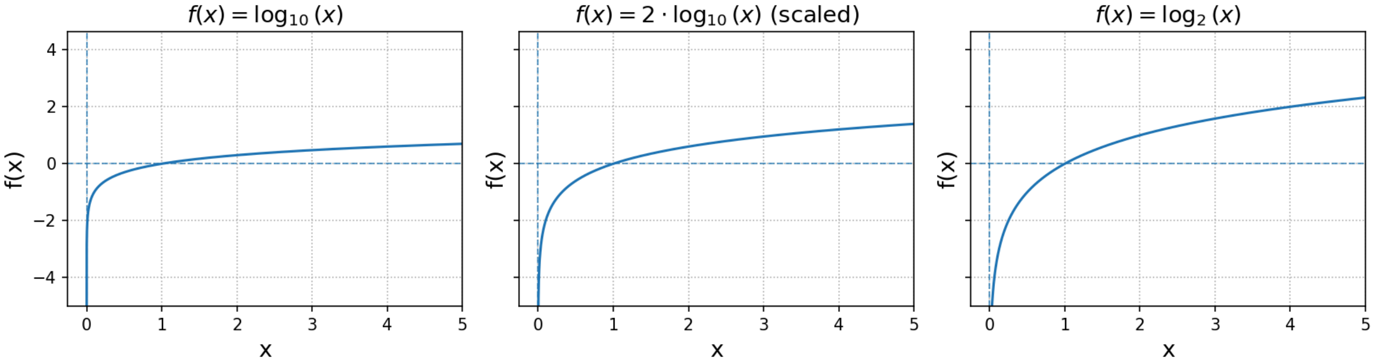
The general form of an exponential function:

$$f(x) = ab^x$$

- Domain: All real numbers ($\mathbb{R}$)
- Range:
  - If $a > 0$ it is $(0, \infty)$
  - If $a < 0$ it is $(-\infty, 0)$
- Graph:
  - If $b > 1$ the graph is increasing (growth)
  - If $0 < b < 1$ the graph is decreasing (decay)
- Asymptote: Horizontal at $y = 0$.

Logarithmic Functions

- Definition



Logarithmic functions are the inverses of exponential functions and can be written in the general form:

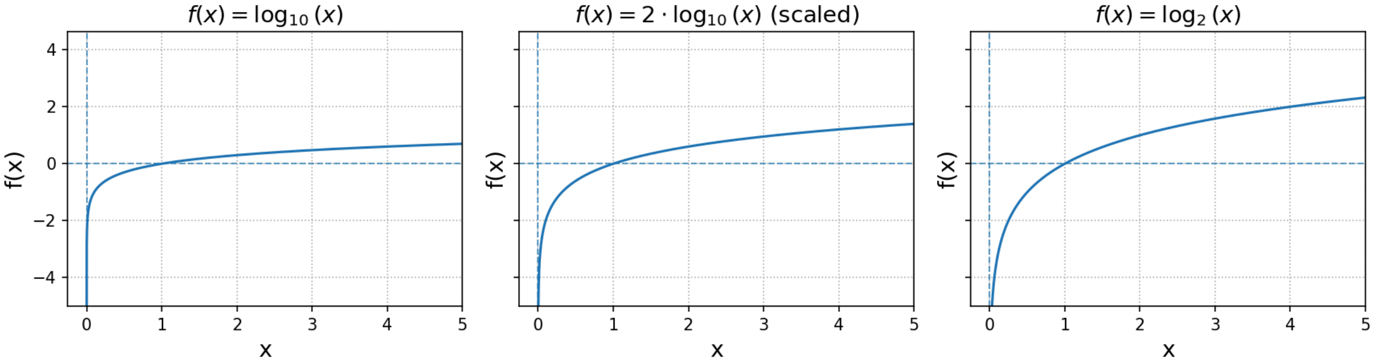
$$f(x) = a \cdot \log_b(x)$$

Here:

- $a \neq 0$
- $b > 0$, and $b \neq 1$

Logarithmic Functions

- Definition



The general form of a logarithmic function:

$$f(x) = a \cdot \log_b(x)$$

- Domain: $(0, \infty)$
- Range: $\mathbb{R}$
- Graph:
  - Passes through $(1, 0)$ if $a = 1$.
  - Slow, unbounded growth for large x.
- Asymptote: Vertical at $x = 0$.

Piecewise Functions

- Definition

A bird's-eye view of piecewise-defined functions:

- Defined by different formulas on different parts of the domain
- Each input still has exactly one output (function rule holds)
- Domain: determined by combined subintervals.
- Range: determined by outputs of all formula pieces.

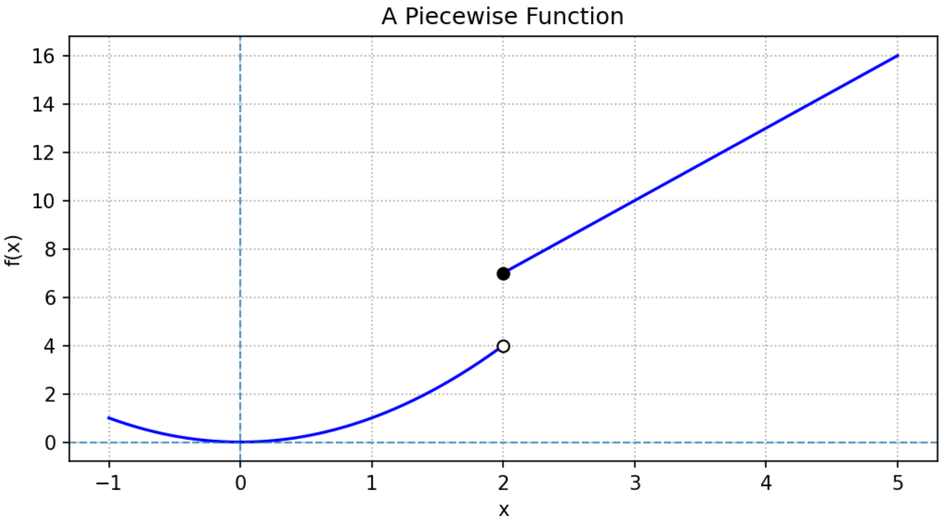
$\rightsquigarrow$ The graph may show breaks or jumps at the boundaries

Piecewise Functions

- Example

A piecewise function might be defined as:

$$f(x) = \begin{cases} 3x + 1 & \text{if } x \geq 2 \\ x^2 & \text{if } x < 2 \end{cases}$$

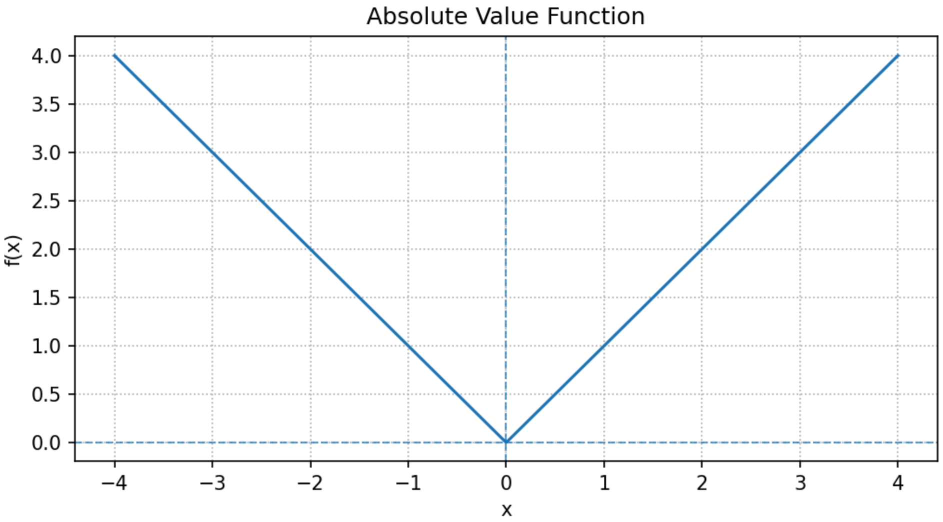


Piecewise Functions

- Example: The Absolute Value Function

The absolute value function:

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

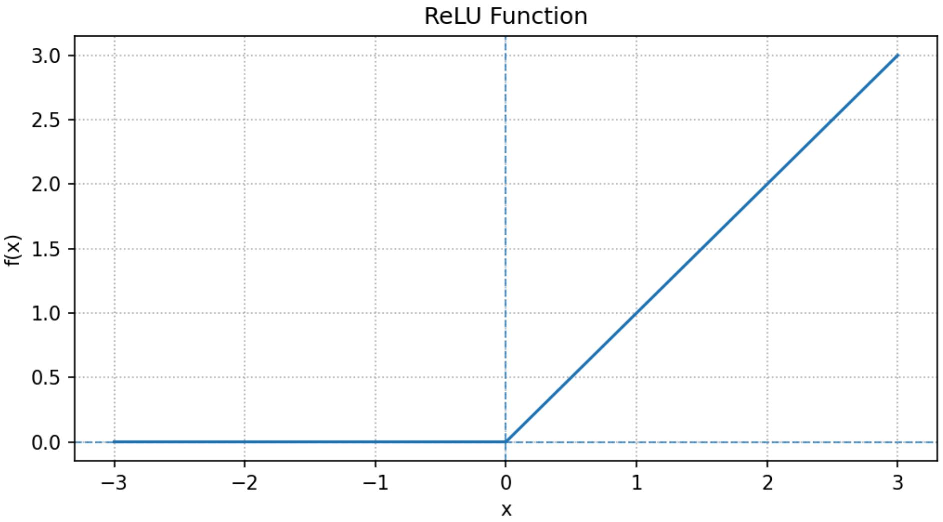


Piecewise Functions

- Example: The ReLU Function

The ReLU activation function (used in neural networks):

$$f(x) = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$



Combining Functions

We can combine functions using standard arithmetic operations.

Given functions f and g (with the same domain), the following rules apply:

Operation	Notation	Definition
Sum	$(f + g)(x)$	$f(x) + g(x)$
Difference	$(f - g)(x)$	$f(x) - g(x)$
Product	$(f \cdot g)(x)$	$f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x)$	$\frac{f(x)}{g(x)}, \quad g(x) \neq 0$

Combining Functions

- Examples

Given $f(x) = x - 1$ and $g(x) = x^2 - 1$, find the functions $(g - f)(x)$ and $\dfrac{g}{f}(x)$:

1. We compute $(g - f)(x)$, as follows:

$$\begin{aligned}(g - f)(x) &= g(x) - f(x) \\ &= x^2 - 1 - (x - 1) = x^2 - x\end{aligned}$$

2. We compute $\frac{g}{f}(x)$, as follows:

$$\begin{aligned}\left(\frac{g}{f}\right)(x) &= \frac{g(x)}{f(x)} \\ &= \frac{x^2 - 1}{x - 1} \\ &= \frac{(x + 1)(x - 1)}{x - 1} = x + 1\end{aligned}$$

Exercise Set

- Part 1

Consider the following functions, what may be appropriate domains, codomain, and range?

1. The battery percentage of a phone is a function of time:
   - Let t be the time (hours since full charge)
   - Let $B(t)$ be the battery level
2. The outcome of an exam is a function of the student's score:
   - Let s be the score
   - Let $O(s)$ the outcome
3. Humidity at a given time of day is a function of time:
   - Let t be the hour
   - $H(t)$ the humidity

Exercise Set

- Part 2

Identify the following functions:

4. $f(x) = 7$
5. $f(x) = \log_2(x)$
6. $f(x) = 3x - 5$
7. $f(x) = \ln(x)$
8. $f(x) = x^2$
9. $f(x) = 5^x$
10. $f(x) = 2x^2 + 1$
11. $f(x) = x(x - 5)$
12. $f(x) = 5 \cdot (0.5)^x$

Let $f(x) = x - 1$ and $g(x) = x^2 - 1$. Find and simplify the functions:

13. $(f \cdot g)(x)$
14. $(f - g)(x)$