Brush-Up Maths for Data Science (2025)

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Basic Algebra

Algebra is the branch of mathematics that:

- Uses symbols and letters to represent numbers (variables)
- Applies tules to manipulate these symbols

In the following, we will explore these ideas in the context of:

- Fractions
- Exponents

Fractions

A fraction has two parts:

$$\frac{\text{Numerator}}{\text{Denominator}}$$

- Numerator: The number above the line
- Denominator: The number below the line, not equal to zero

Fractions represent parts of a whole and are used to express:

- Proportions
- Ratios
- Percentages
- etc...

Fraction Rules

- Adding Fractions

Note that:

$$\frac{a}{b} + \frac{c}{d} \neq \frac{a+c}{b+d}$$

To add two fractions we need to find a common denominator c:

$$\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$$

For example for the 1st appraoch (incorrect):

$$\frac{1+1}{3+3} = \frac{2}{6} = \frac{1}{3}$$

Compared to 2nd approach (correct):

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Fraction Rules

- Adding Fractions (Continued)

The rule of finding a common denominator is based on the formula:

$$\frac{a}{c} + \frac{b}{d} = \frac{a \cdot d + b \cdot c}{c \cdot d}$$

As $c \cdot d$ will be a common denominator of $a \cdot d$ and $b \cdot c$.

For example, using the same fraction as earlier:

$$\frac{1}{3} + \frac{1}{3} = \frac{(1 \cdot 3) + (1 \cdot 3)}{3 \cdot 3} = \frac{6}{9} = \frac{2}{3}$$

Or with different fractions:

$$\frac{1}{4} + \frac{2}{3} = \frac{(1 \cdot 3) + (2 \cdot 4)}{3 \cdot 4} = \frac{11}{12}$$

Fraction Rules

- Multiplying Fractions

Multiplication of fractions is straightforward:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

We simply multiply the numerators and denominators.

For example:

$$\frac{1}{4} \cdot \frac{2}{3} = \frac{1 \cdot 2}{4 \cdot 3} = \frac{2}{12}$$

Fraction Rules

- Dividing Fractions

The division on the other hand is a little more difficult:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

We multiply the first fraction by the reciprocal of the second fraction.

For example:

$$\frac{\frac{1}{4}}{\frac{2}{3}} = \frac{1}{4} \cdot \frac{3}{2} = \frac{1 \cdot 3}{4 \cdot 2} = \frac{3}{8}$$

Exponents

- Products & Powers

Exponents indicate how many times a base number is multiplied by itself, e.g:

$$2^2 = 2 \cdot 2 \quad \text{or} \quad 4^3 = 3 \cdot 3 \cdot 3$$

The Product Rule: When multiplying terms with the same base, we add their exponents:

$$a^n \cdot a^m = a^{n+m}$$

For example: $2^4 \cdot 2^2 \;=\; 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \;=\; 2^6$.

The Power Rule: When raising an exponential term to another power, we multiply the exponents:

$$(a^n)^m = a^{n \cdot m}$$

For example: $(5^3)^2 \;=\; (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5) \;=\; 5^6$.

Exponents

- Negative Exponents & The Quotient Rule

Negative Exponents: A base to a negative exponent is equal to the reciprocal of the base to the positive exponent:

$$a^{-n} = \frac{1}{a^n}$$

For example: $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

The Quotient Rule: When dividing terms with the same base, we subtract their exponents.

$$\begin{aligned} \frac{a^m}{a^n} &= a^m \cdot \frac{1}{a^n} \\ &= a^m \cdot a^{-n} \\ &= a^{m-n} \end{aligned}$$

For example: $\frac{x^5}{x^2} = x^5 \cdot \frac{1}{x^2} = x^5 \cdot x^{-2} = x^{5-2} = x^3$

Note these rules above only applies when $a \neq 0$, as division by zero is undefined.

Exponents

- Roots as Exponents

Roots can be expressed as fractional exponents. In general, the r-th root of a ($\sqrt[r]{a}$) can be written as follows:

$$a^{\frac{1}{r}} = \sqrt[r]{a}$$

For example:

- The positive square root of a ($\sqrt{a}$) is equal to $a^{\frac{1}{2}}$
- The cube root of a ($\sqrt[3]{a}$) is equal to $a^{\frac{1}{3}}$

Note that:

- All the introduced exponent rules apply when the exponent is any number, not just an integer
- It is generally not possible to simplify expressions like:
  - $2^3 \cdot 3^5$ (the bases are different, not related by a common factor)
  - $2^3 + 3^5$ (considering addition instead of multiplication)

Exercise Set 1

- Part 1

1. Simplify: $\frac{2}{5} + \frac{7}{10}$
2. Simplify: $\frac{3}{8} \cdot \frac{4}{9}$
3. Compute: $\frac{\frac{7}{12}}{\frac{14}{9}}$
4. True or false (explain): $\frac{2}{7} + \frac{3}{7} = \frac{5}{14}$
5. Simplify: $(x^3)^4$
6. Simplify: $\frac{5^6}{5^2}$
7. Rewrite with positive exponents only: $\frac{y^{-3}}{y^{-7}}$

Exercise Set 1

- Part 2

8. Express the following as a root: $27^{\frac{2}{3}}$

9. Simplify: $\frac{2^3}{4^{-2}} \cdot \frac{3}{\frac{9}{8}}$

10. If $a = \frac{1}{2}$ and $b = 4^{-1}$ compute $a^{-2} \cdot b^{\frac{3}{2}}$