Brush-Up Maths for Data Science (2025)

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Set Theory

In maths we study:

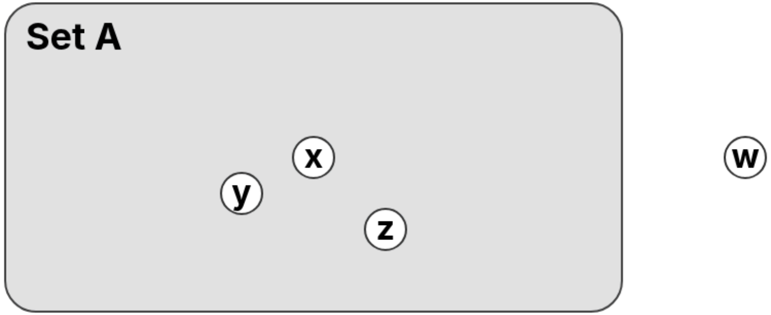
- Patterns, structures, quantities, and logical reasoning

Set theory:

- Is the foundational language of mathematics
- It prvoides a framework for discussing collections of objects

$\rightsquigarrow$ Understanding sets is crucial

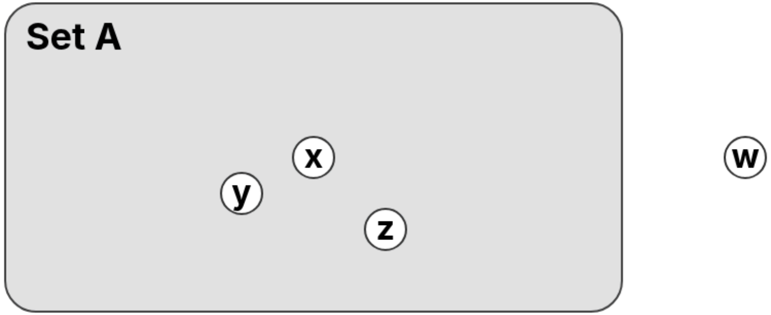
The Definition of a Set



A set is a well-defined collection of distinct objects, called elements.

The Definition of a Set

- Notation



Sets are:

- Typically written using curly braces $\{\}$
- Denoted by capital letters from the Latin alphabet, such as $A = \{x, y, z\}$.

The elements x, y, and z are placeholders and can represent anything from numbers to abstract concepts, as long as they are clearly identifiable.

The Definition of a Set

- Notation (Example)

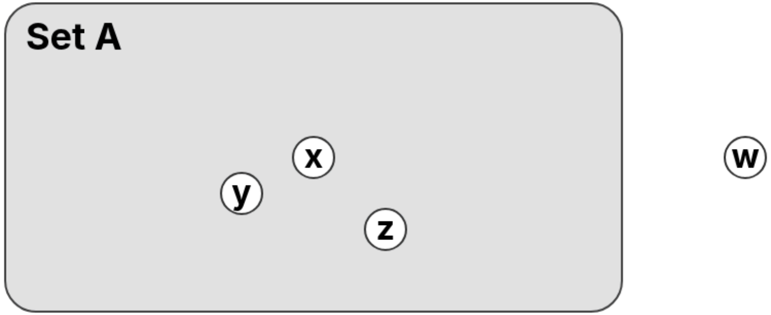
Consider the set of vowels in the English alphabet:

$$A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$$

This set clearly lists all the vowels, and it is easy to determine whether a given letter is a vowel or not.

The Definition of a Set

- Being Well-Defined



The objects inside a set, i.e., its elements, must be well-defined, meaning it is always clear whether something belongs to the set or not.

The Definition of a Set

- Being Well-Defined (Example)

For a set to be well-defined, it must be clear whether any given object is an element of the set or not.

The set of vowels in the word "radio" is well-defined and can be written as:

$$A = \{\mathrm{a}, \mathrm{i}, \mathrm{o}\}$$

Similarly, the following set would also be well-defined:

"All days last year with temperatures below 0° *C"*

because it is based on objective, measurable data.

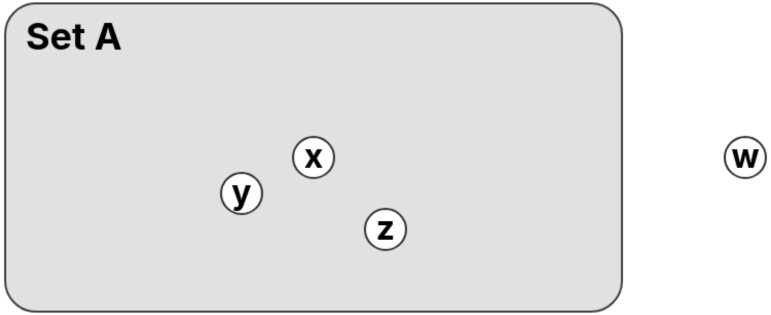
However, the following set would be ill-defined:

"All cold days last year"

because the term "cold" is subjective and can vary from person to person.

The Definition of a Set

- Distinct Elements



A set contains only distinct elements. Duplicates are:

- Not allowed
- Considered the same element if listed more than once

The Definition of a Set

- Distinct Elements (Example)

Suppose we are given the following expression containing vowels of the English alphabet:

$$A = \{\mathrm{a}, \mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$$

This set contains redundant information, because it contains duplicate elements.

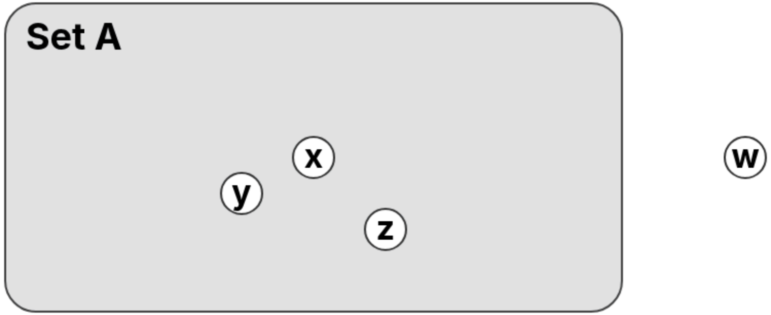
Instead, the proper set of vowels in the English alphabet is:

$$A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$$

Here, each element appears only once.

The Definition of a Set

- Order Independence



The order of elements in a set does not matter.

The Definition of a Set

- Order Independence (Example)

Consider two sets containing the vowels of the English alphabet:

$$A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\} \quad \text{and} \quad B = \{\mathrm{u}, \mathrm{o}, \mathrm{i}, \mathrm{e}, \mathrm{a}\}$$

These two sets are identical because they contain the same elements

In fact, we can write:

$$A = B$$

Noting that the arrangement of elements does not define the uniqueness of a set.

The Empty Set

$$\emptyset = \{ \quad \}$$

The empty set:

- Is the unique set that contains no elements
- It plays a key role in set theory, similar to how 0 plays a key role in arithmetic

Representing Sets

Sets can be described in different ways, depending on the context.

The choice of notation often depends on:

- Whether the set is finite or infinite
- Whether it is discrete or continuous
- How much clarity or formality is needed

We will look at some common methods of representing sets, to understand:

- Their typical use-cases
- Their advantages

Representing Sets

- Verbal Description

Definition:

A verbal description defines a set using written language, explaining its elements or properties.

When to use:

- Providing context before using formal notation

Examples:

1. *"The set of vowels in the English alphabet."*
2. *"The set of whole numbers."*
3. *"The set of whole numbers strictly smaller than 6."*

Representing Sets

- Roster Form

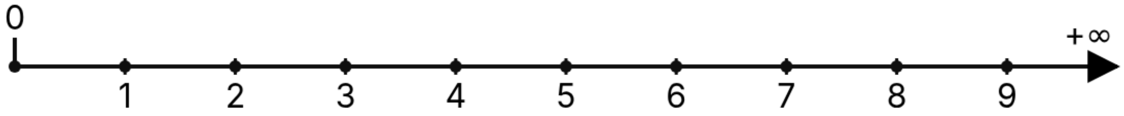
Definition:
Roster form lists all elements of a set inside curly braces $\{\ \}$.

When to use:

- Finite sets
- Infinite sets with a clear pattern

Examples:

1. $A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$. This is the vowels of the english alphabet.
2. $B = \{0, 1, 2, 3, \ldots\}$. This is the set of all whole numbers.
3. $C = \{0, 1, 2, 3, 4, 5\}$. This is the set of whole numbers strictly less than 6.



Representing Sets

- Set-Builder Notation

Definition:

A precise way to define a set by describing the rule its elements must satisfy.

$$\{x \mid \text{condition on } x\} \quad \text{or} \quad \{x : \text{condition on } x\}$$

- x serves as a placeholder representing any potential element of the set
- The bar ($\mid$) or colon ($:$) separates the element from the condition
- The condition determines membership in the set

When to use:

- Ideal for infinite sets or sets with complex conditions

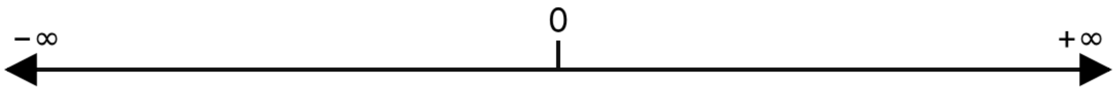
Examples:

1. $A = \{x \mid x \text{ is a vowel in the English alphabet}\}$
2. $B = \{x \mid x \text{ is a whole number}\}$
3. $C = \{x \mid x \text{ is a whole number and } x < 6\}$

Representing Sets

- The Real Numbers & Interval Notation

Certain sets appear so frequently in mathematics that they are assigned dedicated symbols. One of the most fundamental is the set of real numbers, denoted by $\mathbb{R}$.



- The real numbers form an infinite line
- Each point corresponds to a real number
- Segments of the line (intervals) represent collections of real numbers

Interval notation:

- **[ ]** — inclusive bounds
- **( )** — exclusive bounds

Representing Sets

- The Real Numbers & Interval Notation (Examples)

1. The real numbers strictly between 0 and 1:

$$(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$$

2. The real numbers between 2 and 5, including both:

$$[2,5] = \{x \in \mathbb{R} \mid 2 \leq x \leq 5\}$$

3. The real numbers greater than 3:

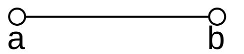
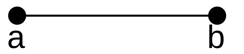
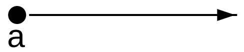
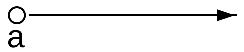
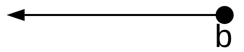
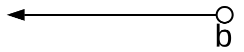
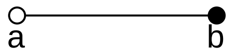
$$(3,\infty) = \{x \in \mathbb{R} \mid x > 3\}$$

4. The real numbers less than or equal to 0:

$$(-\infty,0] = \{x \in \mathbb{R} \mid x \leq 0\}$$

Representing Sets

- The Real Numbers & Interval Notation (Continued)

Set	Interval Notation	Set-Builder Notation	Illustration
Open interval	(a, b)	$\{x \mid a < x < b\}$	
Closed interval	$[a, b]$	$\{x \mid a \leq x \leq b\}$	
Infinite to the right	$[a, \infty)$	$\{x \mid x \geq a\}$	
Infinite to the right	(a, ∞)	$\{x \mid x > a\}$	
Infinite to the left	$(-\infty, b]$	$\{x \mid x \leq b\}$	
Infinite to the left	$(-\infty, b)$	$\{x \mid x < b\}$	
Half-open (left open)	$(a, b]$	$\{x \mid a < x \leq b\}$	

Exercise Set 1

- Part 1

Write the following sets in roster form (if possible). If it is not possible, explain why.

1. The set of first five positive and even whole numbers.
2. The real numbers in the interval $(2, 3)$
3. The set of whole numbers less than 6.
4. The set of whole numbers that are considered very small.
5. The set of letters in the word "banana".
6. The set of even numbers that are typically interesting.
7. The real numbers in the interval $[2, 2]$

Use set-builder notation to describe the following sets:

8. $\{1, 2, 3, 4, 5, 6, 7\}$
9. $\{1, 10, 100, 1000, 10000\}$
10. $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$
11. $[7, 7]$, $(7, 7)$, $(7, 7]$ and $[7, 7)$

Exercise Set 1

- Part 2

Use interval notion to describe the following sets:

12. The set of all real numbers between 2 and 5, including both endpoints.
13. The set of all real numbers strictly greater than -1.
14. The set of all real numbers less than or equal to 4.
15. The set of all real numbers greater than 0 and less than or equal to 10.
16. The empty set (in terms of intervals).

Set Relations

Understanding how elements relate to sets is fundamental, not only when defining a single set, but also when comparing and working with multiple sets.

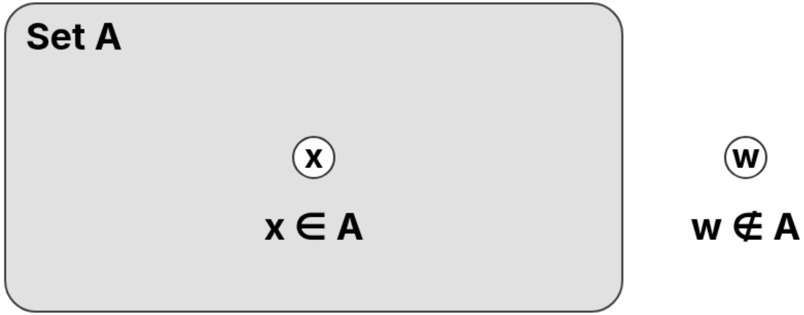
In this section, we introduce the following key relations and their common symbols:

- **Membership** ($\in$, $\notin$)
- **Equality** ($=$, $\neq$)
- **Cardinality** ($|A|$)
- **Subsets** ($\subseteq$, $\subset$)
- **Set difference** ($\setminus$)
- **Complement** (A')

These relations help describe and quantify how sets compare to each other in terms of their elements.

Set Membership

- Definition



Consider elements x and w, we then write:

- $x \in A$: Since x is a member of A
- $w \notin A$: Since w is not a member of A

Set Membership

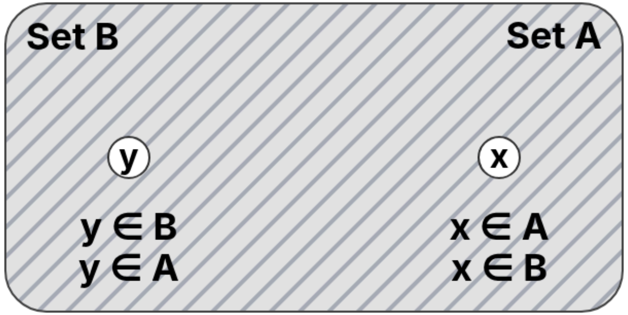
- Examples

The concept of set membership lets us concisely make and asses statements like:

1. Is $\frac{7}{4} \in [2, \sqrt{7})$?
2. Is $\{1\} \in A,\ A = \{1, 2, 3\}$
3. If $x \in (0, \infty)$ then x is positive

Equality of Sets

- Definition



Two sets A and B are equal ($A = B$) if they contain exactly the same elements:

- Every element of A is in B
- Every element of B is in A

For example, if $A = \{x, y\}$ and $B = \{x, y\}$, we can check that $x, y \in A$ and $x, y \in B$, meaning $A = B$.

Cardinality

- Definition & Examples

Definition

The cardinality of a set A, written $|A|$, indicates the number of elements in A.

Examples:

1. Let $A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$, then $|A| = 5$.
2. The empty set $\emptyset$ has cardinality 0.
3. Let $S = \{1, 2, 3\}$ and $T = \{5, 3, 8\}$, then $|S| = |T| = 3$ even though $S \neq T$.

Set Relations

- A Note on Equivalence vs. Equality

Equality ($A = B$):

- Both sets have exactly the same elements.

Equivalence ($|A| = |B|$):

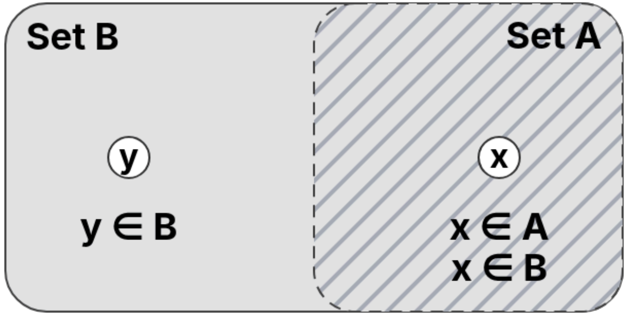
- Sets have the same number of elements (same cardinality).

Key points:

- If $A = B$, then it must be the case that $|A| = |B|$.
- If $|A| = |B|$, it does not follow that $A = B$.

A Subset

- Definition



A is a subset of B ($A \subseteq B$) if every element of A is in B.

For example, if $A = \{x\}$ and $B = \{x, y\}$, then since $x \in A$ and $x \in B$, then $A \subseteq B$.

A Subset

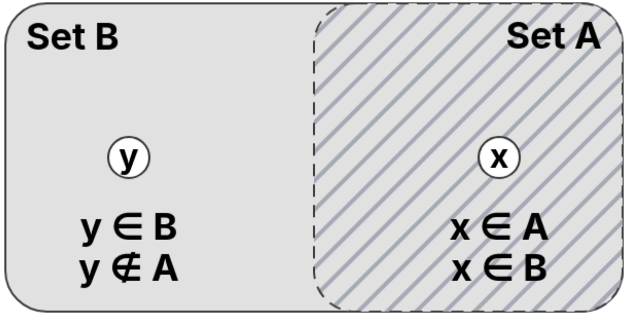
- Examples

In each case, determine if S a subset of T or vice versa:

1. Let $S = \{3, 5, 8\}$ and $T = \{5, 3, 8\}$
2. Let $S = \{\text{red}, \text{blue}\}$ and $T = \{\text{red}, \text{blue}, \text{green}\}$
3. Let $S = \emptyset$ and $T = \{5, 3, 8\}$
4. $S = \{3, 5, 8\}$ and $T = \{5, 8, 3, 2, 6\}$

A Proper Subset

- Definition



A is a proper subset of B ($A \subset B$) if:

- $A \subseteq B$, and
- $A \neq B$ (contains fewer elements)

For example, if $A = \{x\}$ and $B = \{x, y\}$, then since $y \notin A$ but $y \in B$, then $A \subset B$.

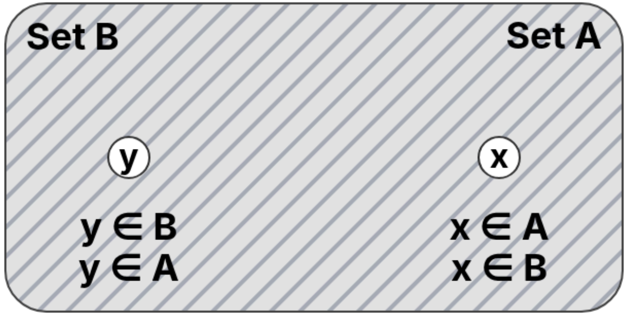
A Proper Subset

- Examples

In each case, determine if S a proper subset of T or vice versa:

1. Let $S = \{3, 5, 8\}$ and $T = \{5, 3, 8\}$
2. Let $S = \{\text{red}, \text{blue}\}$ and $T = \{\text{red}, \text{blue}, \text{green}\}$
3. Let $S = \emptyset$ and $T = \{5, 3, 8\}$
4. $S = \{3, 5, 8\}$ and $T = \{5, 8, 3, 2, 6\}$

Equality & Subset



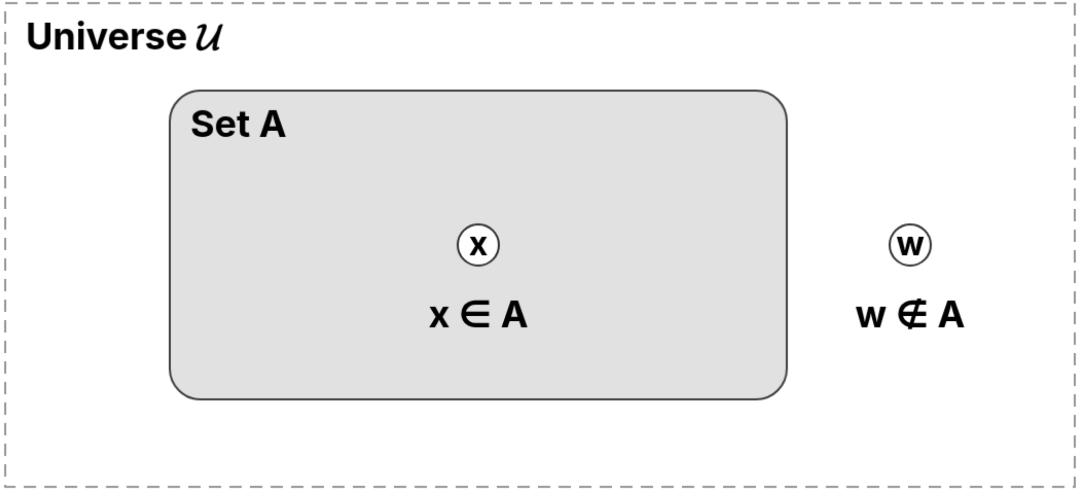
If $A = B$, then:

- $A \subseteq B$
- $B \subseteq A$

Every element of one set is in the other.

The Universe

- Definition



The universe ($\mathcal{U}$) is the set containing all elements under consideration in a given context.

- All sets are considered subsets of $\mathcal{U}$

Exercises Set 2

- Part 1

For each pair of sets below, determine whether they are:

- equal
- equivalent but not equal
- neither equal nor equivalent

1. Let $P = \{1, 2, 3, 4\}$ and $Q = \{3, 2, 1, 4\}$
2. Let $P = \{1, 2, 3\}$ and $Q = \{a, b, c\}$
3. Let $P = \{\{1, 2\}, \{1, 3\}\}$ and $Q = \{\{1, 2\}, \{3, 1\}\}$
4. Let $P = \{\}$ and $Q = \{\emptyset\}$
5. Let $P = \{\{1, 2\}\}$ and $Q = \{\{1, 2\}, \{2, 1\}\}$

Exercises Set 2

- Part 2

Let $X = \{0, 1, 2\}$, $Y = \{1, 2\}$, and $Z = \{1, 2, 3\}$.

State whether each is true, false, or meaningless:

5. $1 \in X$
6. $\{1\} \in X$
7. $Y \subseteq X$
8. $X \subseteq Z$
9. $\emptyset \in Z$
10. $\emptyset \subseteq Z$

The Universe

- Examples

If we let the universe be the set of all English letters:

$$\mathcal{U} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}, \ldots, \mathrm{z}\}$$

Then it gives context to the following sets:

- $A = \{x \mid x \text{ is a vowel}\}$
- $B = \{x \mid x \text{ is a consonant}\}$

If we let the universe be the set of real numbers:

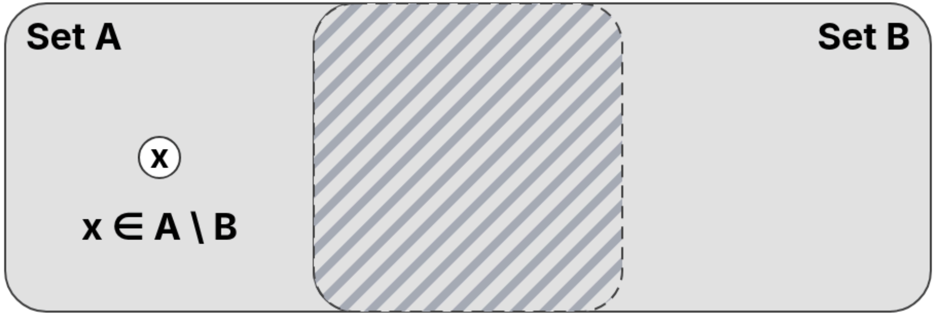
$$\mathcal{U} = \mathbb{R}$$

Then it gives context to the following sets:

- The numbers strictly between 0 and 1: $(0, 1)$
- The numbers between 2 and 5, including both: $[2, 5]$
- The real numbers greater than 3: $(3, \infty)$

Set Difference

- Definition



The difference of A and B, written $A \setminus B$, is:

$$A \setminus B = \{x \in A \mid x \notin B\}$$

That is, we obtain the elements in A but not in B.

Set Difference

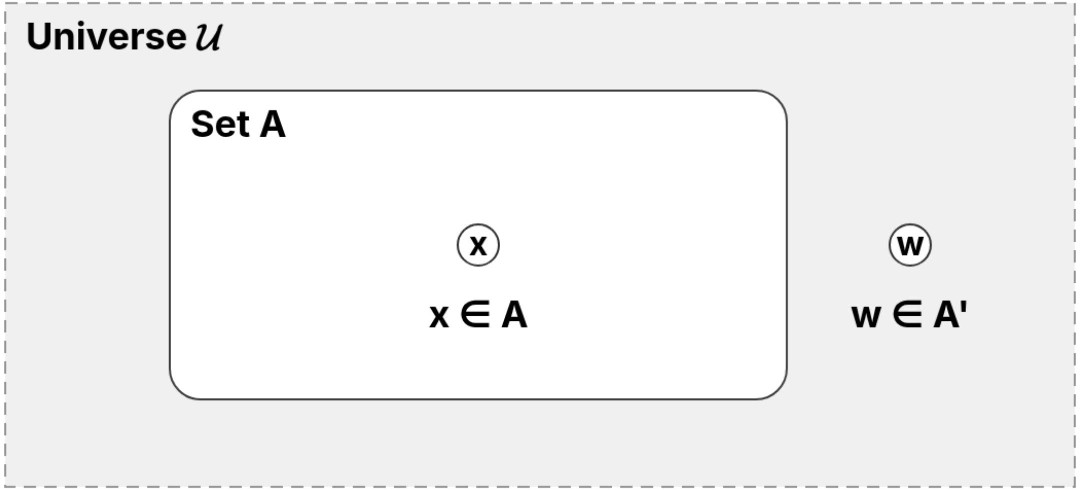
- Examples

In each case compute $A \setminus B$:

1. Let $A = \{1, 2, 3\}$ and $B = \{2, 4\}$
2. Let $A = \{\text{apple}, \text{banana}, \text{pear}\}$ and $B = \{\text{pear}, \text{peach}\}$
3. Let $A = \{1, 2\}$ and $B = \{1, 2\}$
4. Let $A = [0, 1]$ and $B = (0, 1)$

The Complement

- Definition



The complement of A, denoted A', is:

$$A' = \mathcal{U} \setminus A$$

All elements in the universe $\mathcal{U}$ that are not in A

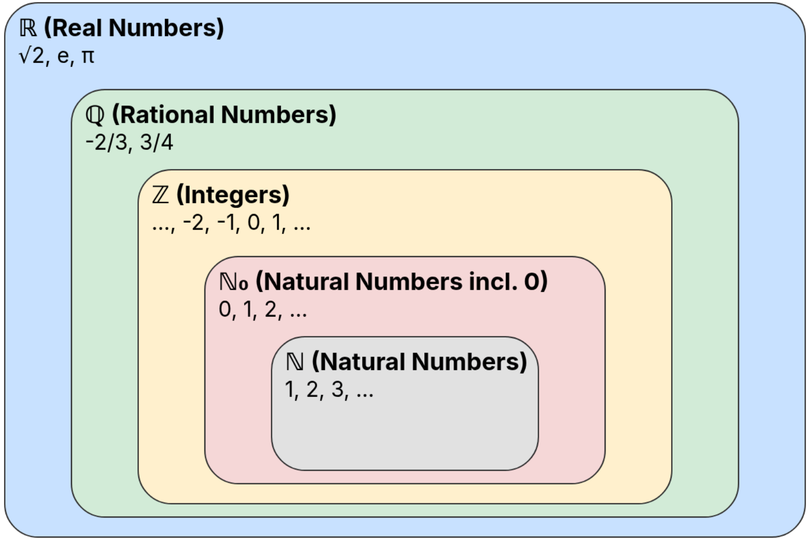
The Complement

- Examples

In each case, given a universe $\mathcal{U}$, determine A':

1. Let $\mathcal{U} = \{1, 2, 3, 4, 5\}$ and $A = \{1, 2, 3\}$
2. Let $\mathcal{U}$ be a deck of cards and A the set of spades
3. If $\mathcal{U} = \mathbb{R}$ and $A = \{x \in \mathbb{R} \mid x > 10\}$.

Special Number Sets



Certain sets of numbers frequently appear and are often represented by special symbols, ranging from the most basic counting numbers to the broadest set of real numbers.

These sets are not isolated; rather, they form a natural hierarchy where smaller sets are contained within larger ones.

Exercise Set 3

List all elements of the following sets:

1. $\{x \in \mathbb{N}_0 \mid x \leq 5\}$
2. $\{x \in \mathbb{R} \mid x^2 = 16\}$
3. $\{x \in \mathbb{Z} \mid -2 \leq x \leq 2\}$
4. $\{x \in \mathbb{Z} \mid x = x + 1\}$

Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{2, 4, 6\}$, and $B = \{1, 2, 3, 4\}$.

Find each of the following sets:

5. $A \setminus B$
6. $B \setminus A$
7. A'
8. B'
9. $(A \setminus B) \setminus A'$

Additional Set Operations

A few other foundational set operations are commonly used in mathematics. Table below provides a brief overview.

Symbol	Operation	Description
$A \cap B$	Intersection of A and B	The set of all elements that are in both A and B
$A \cup B$	Union of A and B	The set of all elements that are in A, in B, or in both
$A \times B$	Cartesian product of A and B	The set of all ordered pairs (a, b) where $a \in A$ and $b \in B$
$\mathcal{P}(A)$	The power set	The set of all subsets of A, including the empty set ($\emptyset$) and A itself