Mathematics Brush-Up for Data Science

Lecture Slides 9: Limits

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Limits

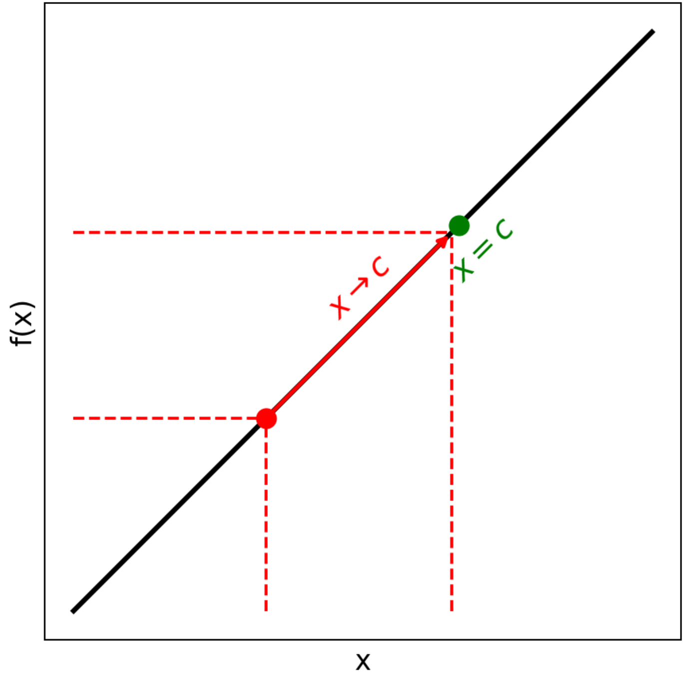
The concept of a limit concerns the value that a function $f(x)$ approaches as $x \to c$.

Note that:

- What happens exactly at c is not what matters
- What happens around the point c is what does

Why understand limits?

- Fundamental to study the continuity of functions
- Limits are the foundation for defining derivatives



Limits

- Building Intuition

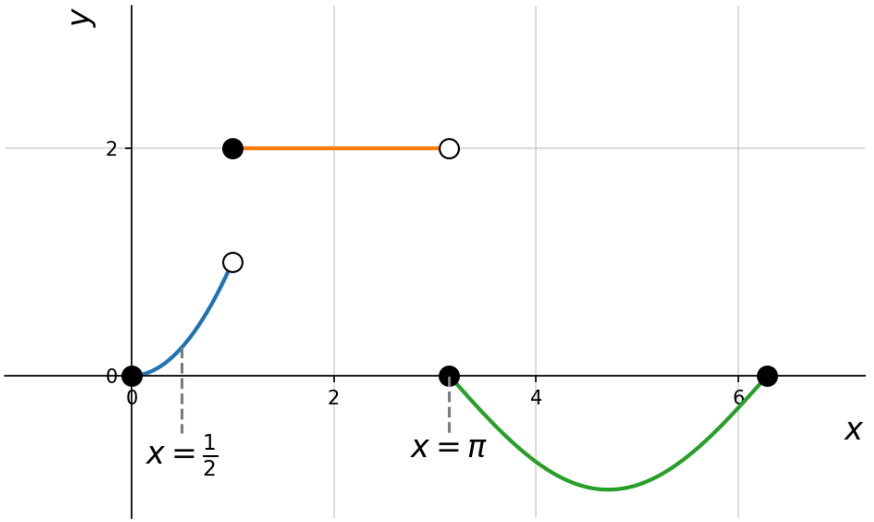
To illustrate and build an intuitive understanding, consider the function and its corresponding graph:

$$f(x) = \begin{cases} x^2, & x \in [0, 1) \\ 2, & x \in [1, \pi) \\ \sin(x), & x \in [\pi, 2\pi] \end{cases}$$

Let us consider the limit of $f(x)$ as $x \to \frac{1}{2}$:

- Look at values of $f(x)$ for x around $\frac{1}{2}$.
- Examine the function x^2 on the interval $[0, 1)$.

As x approaches $\frac{1}{2}$ from either the left or the right, the value of $f(x)$ approaches $\frac{1}{4}$.



The limit of $f(x)$ as $x \to \frac{1}{2}$ can be expressed as:

$$\lim_{x \to \frac{1}{2}} f(x) = \frac{1}{4}$$

Limits

- Definition

To more formally define the limit of a function, we let:

- f be defined on an open interval containing c, possibly except at c
- c and L be real numbers

In this case:

- As x gets closer to c (but is not equal to c)
- If $f(x)$ gets closer and stays close to L

$\rightsquigarrow$ We say: The limit of $f(x)$ as x approaches c is L

Symbolically, we express this idea as:

$$\lim_{x \to c} f(x) = L$$

if and only if, it is the case that:

$$\lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L$$

Note:

- $x \to c^+$: approaching from the right, with $x > c$
- $x \to c^-$: approaching from the left, with $x < c$

Limits

- When They Do Not Exist

Note that:

- Not every function has a limit at every point
- A limit may fail to exist for several reasons

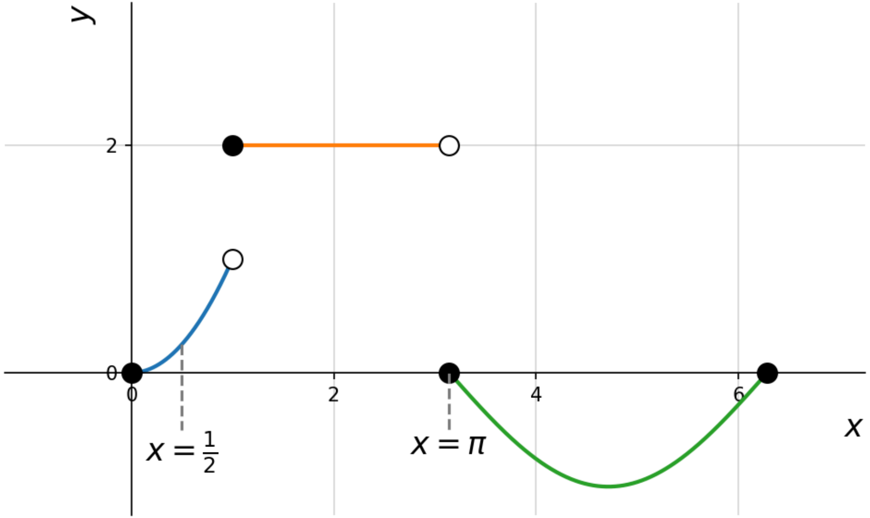
To illustrate this, consider:

- The earlier function and its graph
- The point $x = \pi$

The left-hand and right-hand limits differ:

$$L^- = \lim_{x \to \pi^-} f(x) = 2 \quad \text{and} \quad L^+ = \lim_{x \to \pi^+} f(x) = 0$$

$\rightsquigarrow \lim\limits_{x \to \pi} f(x)$ does not exist. The same reasoning also applies to the point $x = 1$



Another way a finite limit can fail to exist is if the function grows without bound. For example: $\lim_{x \to \infty} x^2$

As $x \to \infty$, then $x^2 \to \infty$, i.e., the function does not approach a finite value.

Limits at Infinity and Infinite Limits

- Two Different Behaviors

When the input grows without bound, the function may still approach a finite value:

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

As x grows, the output approaches 0.

By contrast, the output may grow without bound near a finite input:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty, \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

The two one-sided behaviors differ, so the two-sided limit does not exist.

$\rightsquigarrow$ The first limit has a finite value; the second describes unbounded output rather than a finite limit.

Exercise Round 1

- Limit Intuition

Determine each limit intuitively, using a small table or graph if helpful:

1. $\lim_{x \to 2} 2x$
2. $\lim_{x \to \infty} x$
3. $\lim_{x \to \infty} \left(3 + \frac{1}{x}\right)$
4. $\lim_{x \to 0} \frac{1}{x^2}$

Discontinuities

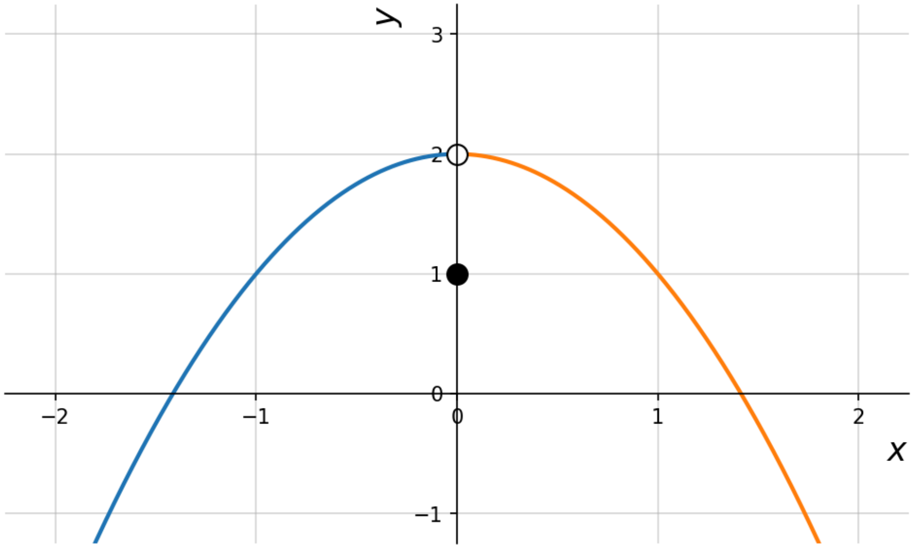
A function is discontinuous at a point if it fails to be continuous there.

In our example, f has discontinuities at $x = 1$ and $x = \pi$ because the limits at these points do not exist.

However, a discontinuity does not always mean the limit fails to exist.

For instance, consider the function and its graph:

$$f(x) = \begin{cases} -x^2 + 2, & x \in \mathbb{R} \setminus \{0\} \\ 1, & x = 0 \end{cases}$$



It is clear that despite $\lim_{x \to 0} f(x) = 2$, the exact function value at $x = 0$ is $f(0) = 1$

$\rightsquigarrow$ We say that f is discontinuous at $x = 0$.

Continuity

- Definition

To define the continuity of a function, let:

- $f : A \to \mathbb{R}$ be a function
- $A \subseteq \mathbb{R}$ the domain of f

In this case, f is said to be continuous at $c \in A$ if:

$$\lim_{x \to c} f(x) = f(c)$$

Breaking it down:

1. The limit exists:
$$\lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L$$
2. The function value $f(c)$ is defined
3. The limit equals the function value: $L = f(c)$

Continuity is checked point by point. If f is continuous at every $c \in A$, then f is continuous on A.

At an interior point, use the two-sided limit condition shown on the left.

At an endpoint, approach using values from within the domain:

$$\begin{aligned} \text{left endpoint } a : &\quad \lim_{x \to a^+} f(x) = f(a), \\ \text{right endpoint } b : &\quad \lim_{x \to b^-} f(x) = f(b). \end{aligned}$$

Most common functions are continuous throughout their natural domains.

Continuity

- Example: A Removable Discontinuity

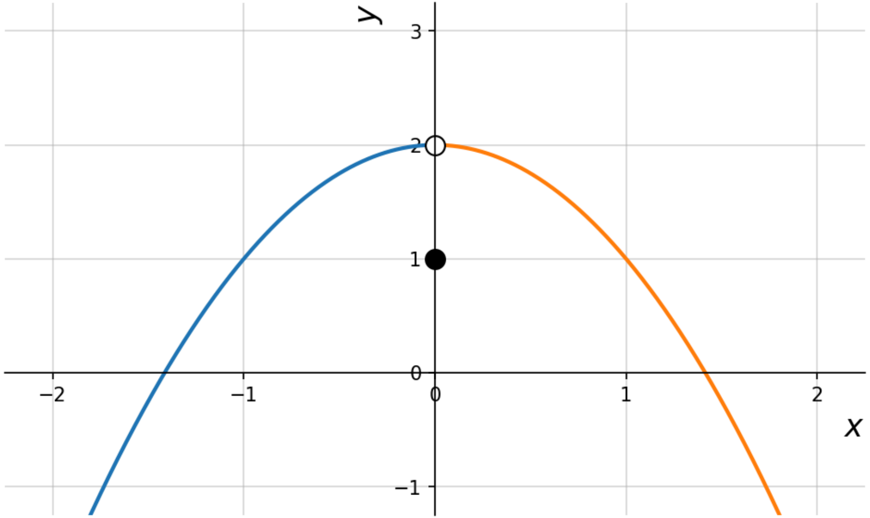
Recall the earlier function and its graph:

$$f(x) = \begin{cases} -x^2 + 2, & x \in \mathbb{R} \setminus \{0\} \\ 1, & x = 0 \end{cases}$$

Is the function continuous at $x = 0$?

We can break down the continuity condition:

1. The limit exists:
$$\lim_{x \to 0^+} f(x) = 2 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = 2 \quad \checkmark$$
2. The function value $f(0) = 1$ is defined $\checkmark$
3. The limit does not equal the function value:
$$\lim_{x \to 0} f(x) = 2 \neq f(0) = 1 \quad \times$$



Exercise Round 2

- Continuity

Consider the piecewise function

$$g(x) = \begin{cases} 2x+1, & x<1, \\ 2, & x=1, \\ -x+4, & x>1. \end{cases}$$

1. Is g continuous at $x = 1$? If not, which continuity condition fails?

Limit Laws

Earlier, we combined functions using arithmetic operations.
Assuming both limits exist, the same operations obey these laws:

Operation	Limit law
Constant multiple	$\lim_{x \to c}[k \cdot f(x)] = k \cdot \lim_{x \to c} f(x)$
Sum	$\lim_{x \to c}[f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$
Difference	$\lim_{x \to c}[f(x) - g(x)] = \lim_{x \to c} f(x) - \lim_{x \to c} g(x)$
Product	$\lim_{x \to c}[f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x)$
Quotient	$\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}, \quad \lim_{x \to c} g(x) \neq 0$

When f is continuous at c, direct substitution gives $\lim_{x \to c} f(x) = f(c)$.

Limit Laws

- Examples: Direct Substitution and Factoring

Determine $\lim_{x\to 1} \left(\sqrt{x+3} - \sqrt{4-x}\right)$

1. Split into two limits (use the difference law):
$$\lim_{x\to 1} \sqrt{x+3} - \lim_{x\to 1} \sqrt{4-x}$$

2. Evaluate each part directly (direct substitution):
$$\begin{aligned} \lim_{x\to 1} \sqrt{x+3} - \lim_{x\to 1} \sqrt{4-x} &= \sqrt{1+3} - \sqrt{4-1} \\ &= \sqrt{4} - \sqrt{3} \\ &= 2 - \sqrt{3} \end{aligned}$$

Determine $\lim_{x\to 1} \frac{x^2-1}{x-1}$

1. Check direct substitution (does not work!):
$$\frac{1^2-1}{1-1} = \frac{0}{0}$$

2. Factor numerator and cancel the common factor:
$$\begin{aligned} \lim_{x\to 1} \frac{x^2-1}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+1)}{x-1} \\ &= \lim_{x\to 1} x+1 \end{aligned}$$

3. Evaluate the expression (direct substitution):
$$\begin{aligned} \lim_{x\to 1} x+1 &= 1+1 \\ &= 2 \end{aligned}$$

Exercise Round 3

- Applying Limit Laws

Determine each limit. Name the limit law or algebraic step you use.

1. $\lim_{x \to 1} (x^2 + x)$
2. $\lim_{x \to 2} (2x^3 - 2e^x)$
3. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$