Mathematics Brush-Up for Data Science

Lecture Slides 8: Functions III: Mappings and Inverses

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Function Mappings

- Properties of Input-Output Relationships

The formula of a function tells us how to calculate its outputs.

Mapping properties tell us how its domain and codomain are connected.

We distinguish three important properties:

- **Injective:** different inputs do not share an output.
- **Surjective:** every element of the codomain is reached.
- **Bijective:** the function is both injective and surjective.

These properties depend on the stated domain and codomain, not only on the formula.

They allow us to determine whether an equation of the form $f(x) = y$ has:

- no solution,
- exactly one solution
- multiple solutions

Function Mappings

- Injective Functions

A function $f : A \to B$ is injective (one-to-one) if it never assigns the same output to two different inputs.

Formally,

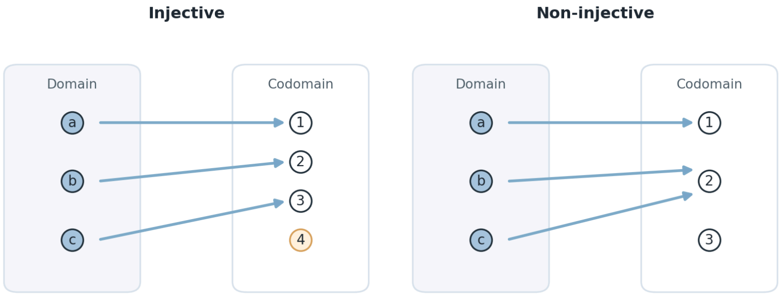
$$f(x_1) = f(x_2) \quad \Rightarrow \quad x_1 = x_2.$$

Each output in B comes from at most one input in A.

For example, the following function is injective:

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = 2x + 3$$

Every output comes from at most one input.



By contrast:

$$g : \mathbb{R} \to \mathbb{R}, \qquad g(x) = x^2$$

is not injective because $g(2) = g(-2) = 4$.

Function Mappings

- Surjective Functions

A function $f : A \to B$ is surjective (onto) if every element of the codomain occurs as an output.

Formally,

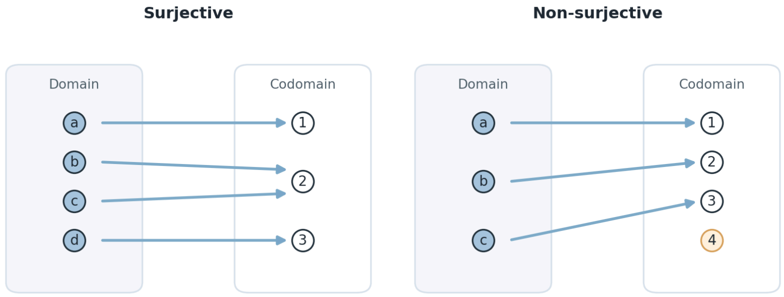
$$\text{For every } y \in B, \text{ there exists } x \in A \text{ such that } f(x) = y.$$

Equivalently, $\mathrm{Range}(f) = B$.

For example, the following function is surjective:

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = 2x + 1$$

Every real target value is reached.



By contrast:

$$g : \mathbb{R} \to \mathbb{R}, \qquad g(x) = x^2$$

is not surjective because it never reaches negative values.

Function Mappings

- Bijective Functions

A function is bijective if it is both injective and surjective.

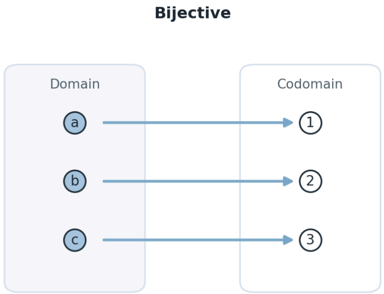
This creates a one-to-one correspondence between A and B:

- no output is repeated, and
- no element of the codomain is left out.

For example, the following function is bijective:

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x + 1$$

Every real output comes from exactly one real input.



Function Mappings

- Comparing the Properties

Property	Can two inputs share an output?	Must every codomain value be reached?
Injective	No	No
Surjective	Yes	Yes
Bijective	No	Yes

Why bijectivity matters:

- Injectivity ensures an output does not point back to several inputs.
- Surjectivity ensures every value in the codomain can be mapped back.
- Together, they make a two-sided inverse function possible.

Exercise Round 1

- Mapping Properties

For each function:

- determine whether it is injective, surjective, bijective, or none of these
- justify your answer using its domain, codomain, and outputs.

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = 2x - 3.$
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = x^2.$
3. $h : [0, \infty) \to [0, \infty),\ h(x) = x^2.$
4. $k : \mathbb{R} \to (0, \infty),\ k(x) = e^x.$

Inverse Functions

- Definition

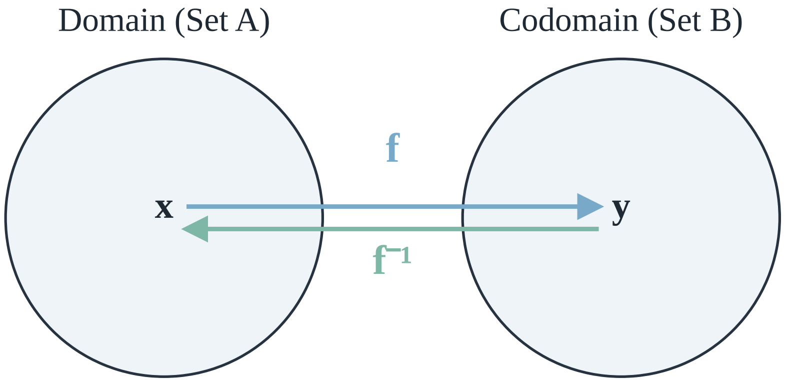
An inverse reverses the mapping of f: it maps each output back to the input that produced it.

Let $f : A \to B$ be bijective. Its inverse $f^{-1} : B \to A$ satisfies, for every $x \in A$ and $y \in B$:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(y)) = y$$

The first identity gives the general rule for recovering an input from an output:

$$\begin{aligned} & f(x) = y \\ \iff \quad & f^{-1}(f(x)) = f^{-1}(y) \\ \iff \quad & x = f^{-1}(y). \end{aligned}$$



Inverse Functions

- Example: Finding and Applying an Inverse

Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = 3x + 5.$$

To find f^{-1}, solve for x in terms of y:

$$y = 3x + 5 \quad \iff \quad x = \frac{y-5}{3}.$$

This expression tells us how to recover x from a given output y, so:

$$f^{-1}(y) = \frac{y-5}{3}.$$

Now suppose we want to solve

$$f(x) = 14$$

To isolate x, we apply f^{-1} to both sides:

$$f^{-1}(f(x)) = f^{-1}(14)$$

Since f^{-1} reverses the action of f, the left-hand side simplifies to x:

$$x = \frac{14-5}{3} = 3$$

This is the reason for applying f^{-1} to both sides: it "undoes" f on the side containing x, leaving x alone.

Inverse Functions

- Example: Checking an Inverse Graphically

For

$$f(x) = 3x + 5, \qquad f^{-1}(x) = \frac{x-5}{3},$$

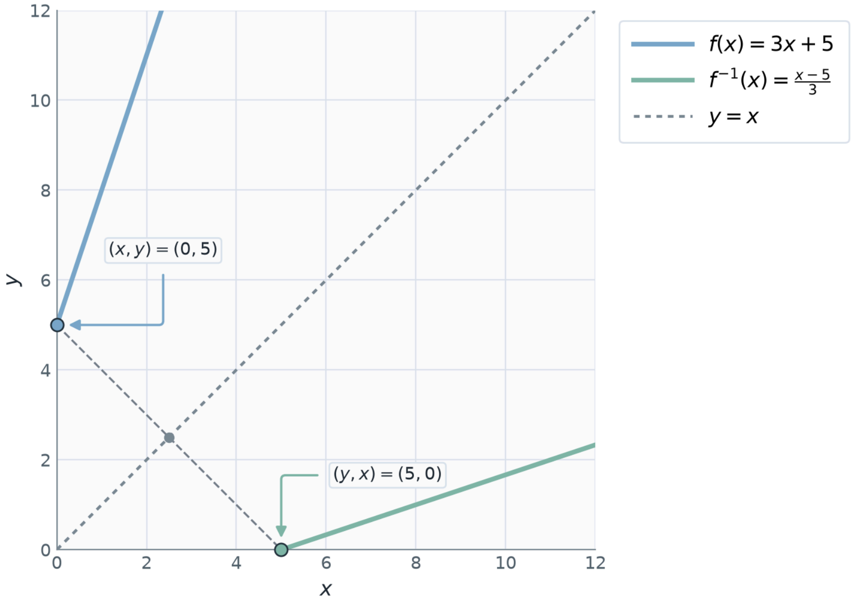
the two graphs are reflections across $y = x$.

For example, each equation determines a point on its graph:

$$f(0) = 5 \quad \Longrightarrow \quad (0, 5),$$

$$f^{-1}(5) = 0 \quad \Longrightarrow \quad (5, 0).$$

The inverse reverses the input-output pair.



Common Inverse Functions

Function $f(x)$	Inverse $f^{-1}(x)$	Domain of f	Range of f
$f(x) = x + a$	$f^{-1}(x) = x - a$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = x - a$	$f^{-1}(x) = x + a$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = kx, k \neq 0$	$f^{-1}(x) = \frac{x}{k}$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = \frac{x}{k}, k \neq 0$	$f^{-1}(x) = kx$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = x^n, n = 1, 3, 5, \ldots$	$f^{-1}(x) = \sqrt[n]{x}$	$\mathbb{R}$	$\mathbb{R}$
$f(x) = x^n, n = 2, 4, 6, \ldots$	$f^{-1}(x) = \sqrt[n]{x}$ (principal root)	$[0, \infty)$	$[0, \infty)$
$f(x) = e^x$	$f^{-1}(x) = \ln(x)$	$\mathbb{R}$	$(0, \infty)$
$f(x) = a^x, a > 0, a \neq 1$	$f^{-1}(x) = \log_a(x)$	$\mathbb{R}$	$(0, \infty)$
$f(x) = \ln(x)$	$f^{-1}(x) = e^x$	$(0, \infty)$	$\mathbb{R}$
$f(x) = \log_a(x), a > 0, a \neq 1$	$f^{-1}(x) = a^x$	$(0, \infty)$	$\mathbb{R}$

The stated domains are essential: for example, an even power becomes one-to-one only after its domain is restricted to $[0, \infty)$.

Exercise Round 2

- Inverse Functions

Find each inverse and state its domain and range. Verify one answer by composing the function with its proposed inverse in both orders.

1. $p(x) = 3x - 5,\ p : \mathbb{R} \to \mathbb{R}.$

2. $q(x) = e^{2x},\ q : \mathbb{R} \to (0, \infty).$

Logarithm Rules

- Key Identities

For $x > 0, y > 0, n \in \mathbb{R}$, and valid logarithm bases $b, k > 0$ with $b, k \neq 1$, the most important rules are given in the following table.

Rule	Formula	Description
Product Rule	$\log_b(xy) = \log_b(x) + \log_b(y)$	The logarithm of a product equals the sum of the logarithms.
Quotient Rule	$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$	The logarithm of a quotient equals the difference of the logarithms.
Power Rule	$\log_b(x^n) = n\log_b(x)$	A power in the argument becomes a multiplier in front of the logarithm.
Logarithm of 1	$\log_b(1) = 0$	Any base raised to the power 0 equals 1.
Logarithm of the Base	$\log_b(b) = 1$	Any base raised to the power 1 equals itself.
Inverse Property	$b^{\log_b(x)} = x$	Exponential and logarithmic functions cancel each other.
Natural Log of e	$\ln(e) = 1$	Since $\ln(x)$ means $\log_e(x)$.
Change of Base	$\log_b(x) = \frac{\log_k(x)}{\log_k(b)}$	Converts a logarithm from one base to another.

Note: The natural logarithm $\ln(x)$ is simply $\log_e(x)$, where $e \approx 2.71828$. All these rules work the same way for $\ln$ as for $\log_b$.

Solving Non-Linear Equations

- An Application of Inverse Functions

What are non-linear equations?

- Polynomial equations of degree at least 2
- Trigonometric and hyperbolic functions
- Exponentials and logarithms
- other non-linear function classes

How do we solve them?

- Transform the equation to isolate the variable
- Check domain restrictions to validate the solution

Key strategy:

- Undo the operation affecting the variable

We will focus on non-linear equations involving:

- Exponentials
- Logarithms

With these equations, the typical solution procedure is:

- If the variable is in the exponent:
  - $\rightsquigarrow$ apply a logarithm to both sides
- If the variable is inside a logarithm:
  - $\rightsquigarrow$ apply an exponential to both sides

Solving Non-Linear Equations

- Example: Exponential and Logarithmic Equations

Let us solve the following equation for x:

$$e^{2x+1} = w, \quad w > 0$$

We proceed as follows:

$$\begin{aligned} e^{2x+1} &= w \\ \Leftrightarrow \qquad \ln\left(e^{2x+1}\right) &= \ln(w) \\ \Leftrightarrow \qquad 2x+1 &= \ln(w) \\ \Leftrightarrow \qquad x &= \frac{1}{2}\left(\ln(w) - 1\right) \end{aligned}$$

Let us solve the following equation for x:

$$\ln(x^2 - 10) = 6$$

Assuming that $x^2 > 10$, we obtain:

$$\begin{aligned} \ln(x^2 - 10) &= 6 \\ \Leftrightarrow \qquad e^{\ln(x^2-10)} &= e^6 \\ \Leftrightarrow \qquad x^2 - 10 &= e^6 \\ \Leftrightarrow \qquad x^2 &= e^6 + 10 \\ \Leftrightarrow \qquad x &= \pm\sqrt{e^6 + 10} \end{aligned}$$

The $\pm$ symbol indicates that both the positive and negative values are solutions. Both satisfy the original domain restriction $x^2 > 10$.

Exercise Round 3

- Exponential and Logarithmic Equations

In the following non-linear equations, use inverse functions and logarithm rules to solve for x.

Remember to check any domain restrictions.

1. $5^{3x-2} = 5$
2. $\log_5(2x+3) = 2$
3. $e^{2x+1} = 7$
4. $\log_2(x) + \log_2(x-2) = 3$

Determining Whether a Relation is a Function

- Example: Checking Finite Relations Graphically

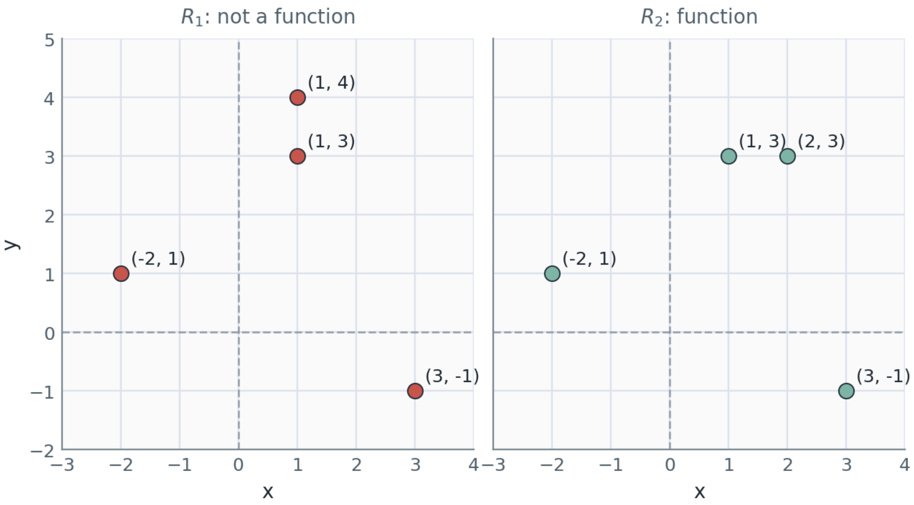
A relation in which each x-coordinate is matched with exactly one y-coordinate is said to describe y as a function of x.

This also means that, if the same x-coordinate is associated with two different y-coordinates, then the relation is not a function.

Example:

Which of the following relations describe y as a function of x?

- $R_1 = \{(-2, 1), (1, 3), (1, 4), (3, -1)\}$
- $R_2 = \{(-2, 1), (1, 3), (2, 3), (3, -1)\}$



Determining Whether a Relation is a Function

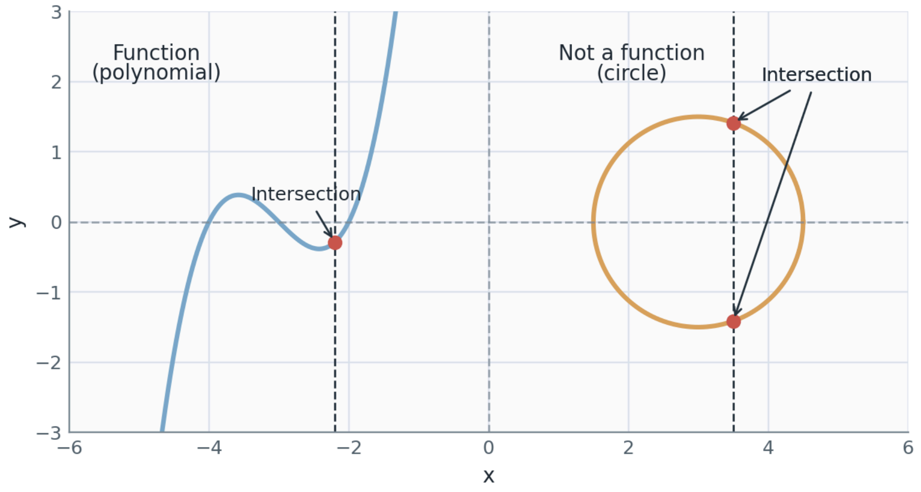
- The Vertical Line Test

This leads to the vertical line test, which is a graphical method to decide whether a relation is a function:

- A relation is a function if and only if every vertical line intersects its graph at most once.
- If a vertical line intersects more than once, the relation assigns more than one output to the same input, thus violating the definition of a function.

Note: Equations can describe valid relationships, like the shape of a circle, but do not define a function.

Recognizing this helps us understand both the limits of function notation and the situations where we need to use other representations.



Determining Whether a Relation is a Function

- Example: Checking a Relation Algebraically

We can check whether an equation defines a function by solving for one variable in terms of the other.

If solving produces more than one output value for the same input, then the relation is not a function.

Example:

Does the equation $x^2 + y^2 = 1$ represent a function with x as input and y as output? If so, express the relationship as a function $y = f(x)$.

Solution:

First we subtract x^2 from both sides:

$$y^2 = 1 - x^2$$

We now try to solve for y in this equation:

$$y = \pm\sqrt{1 - x^2}$$

We get two outputs corresponding to the same input, so this relationship cannot be represented as a single function $y = f(x)$.

Exercise Round 4

- Relations and Functions

Determine whether each equation represents y as a function of x.

1. $x^3 + y^2 = 1$
2. $x^2 + y^3 = 1$
3. $y = \sqrt{4 - x^2}$