Mathematics Brush-Up for Data Science

Lecture Slides 7: Equation Solving

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Equation Solving

Functions have particular points of interest on their graphs due to what they represent:

- The x-intercepts occur where the output value is zero.
- The y-intercept occurs where the input value is zero.

To locate these intercepts:

- For the y-intercept, fix the input at $x = 0$.
- For the x-intercepts:
  - Fix the output at $y = 0$.
  - Determine the corresponding inputs by solving

$$f(x) = 0.$$

This second task requires equation solving.

Generalizing the x-intercept problem:

- Replace the output 0 with a specified output y_0.
- Find the inputs x that satisfy

$$f(x) = y_0.$$

- Graphically, these are the x-coordinates where the graph of f meets $y = y_0$.
- When $y_0 = 0$, they are the x-intercepts.

From Equations to Roots

To solve an equation:

- Find every value of the chosen variable that makes the equality true.
- Rearrange the equation, when possible, to isolate that variable on one side:

$$x = \text{an expression not involving } x.$$

Another useful form is the root-finding form:

- Move all terms to one side.
- Place zero on the other side.

$$f(x) = g(x) \quad \Leftrightarrow \quad f(x) - g(x) = 0.$$

Define the difference function:

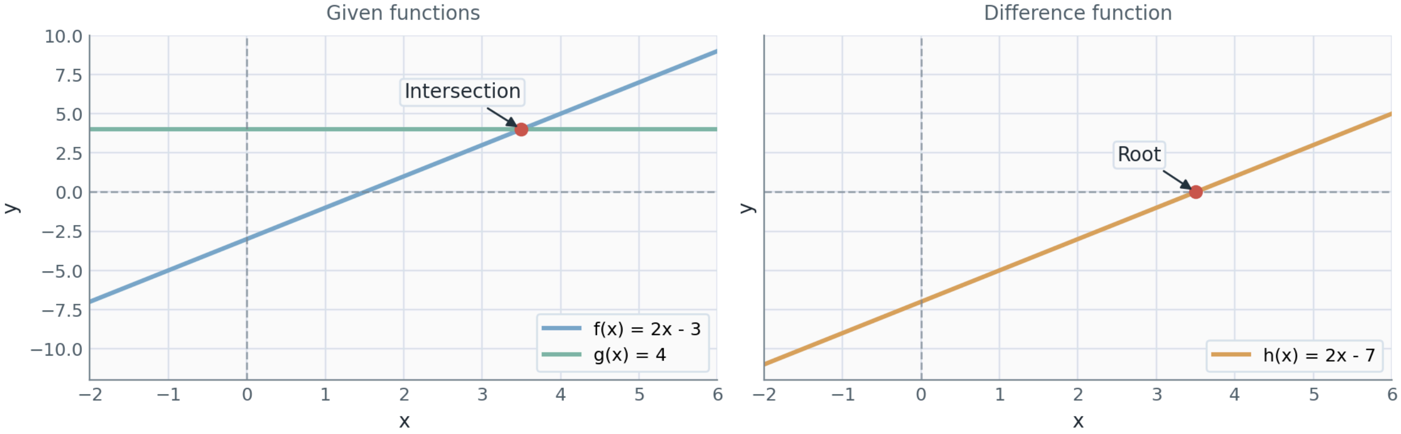
$$h(x) = f(x) - g(x),$$
$$f(x) = g(x) \quad \Leftrightarrow \quad h(x) = 0.$$

A root (or zero) of h is an input r satisfying $h(r) = 0$.

- Its graph point is the x-intercept $(r, 0)$.
- Therefore, solving $f(x) = g(x)$ means finding the roots of h.

Solving Linear Equations

- Example: Linear Equations as Roots



Consider functions:

- $f(x) = 2x - 3$ (linear)
- $g(x) = 4$ (constant)

Finding their intersection means determining x s.t:

$$f(x) = g(x) \quad \Leftrightarrow \quad 2x - 3 = 4$$

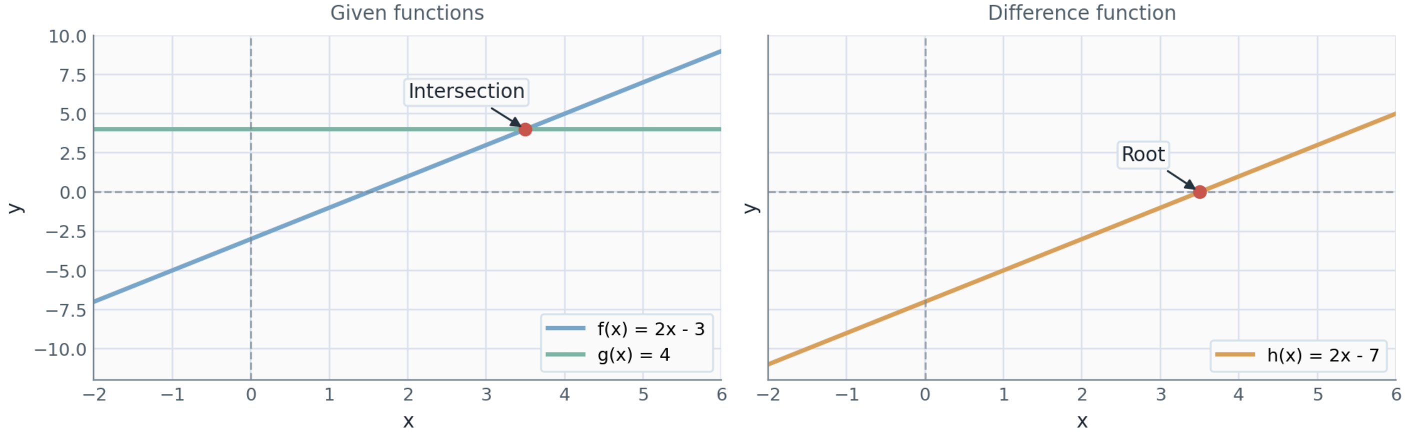
We can convert this to a root-finding problem by moving everything to the left-hand-side (LHS):

$$\begin{aligned} & f(x) - g(x) = 0 \\ \Leftrightarrow \quad & 2x - 3 - 4 = 0 \\ \Leftrightarrow \quad & 2x - 7 = 0 \end{aligned}$$

The LHS can be regarded as a new function $h(x)$.

Solving Linear Equations

- Example: Linear Equations as Roots (Continued)



Finding its root is the same as solving the original equation:

$$
\begin{aligned}
& h(x) = 2x - 7 = 0 \\
\Leftrightarrow \quad & 2x = 7 \\
\Leftrightarrow \quad & \frac{2x}{2} = \frac{7}{2} \\
\Leftrightarrow \quad & x = 3.5
\end{aligned}
$$

$\rightsquigarrow$ The algebraic solution matches the geometric interpretation:

- We found that the solution is $x = 3.5$
- The graphs intersect at the same point

Exercise Round 1

- Linear Equations

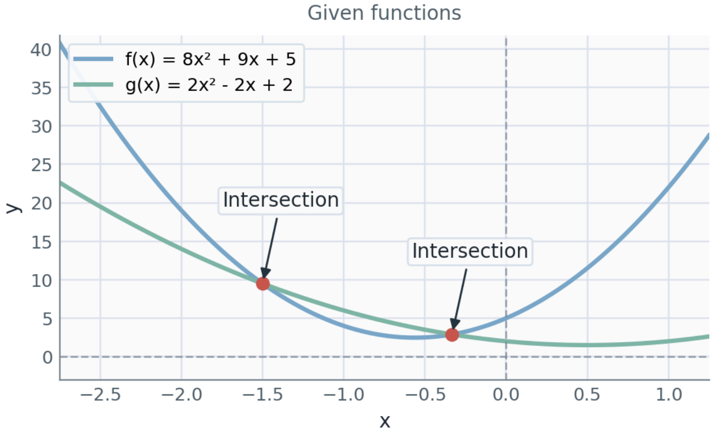
Solve each equation for x, then substitute your result into the original equation to verify it. For each equation, also describe the equivalent root-finding problem $h(x) = 0$.

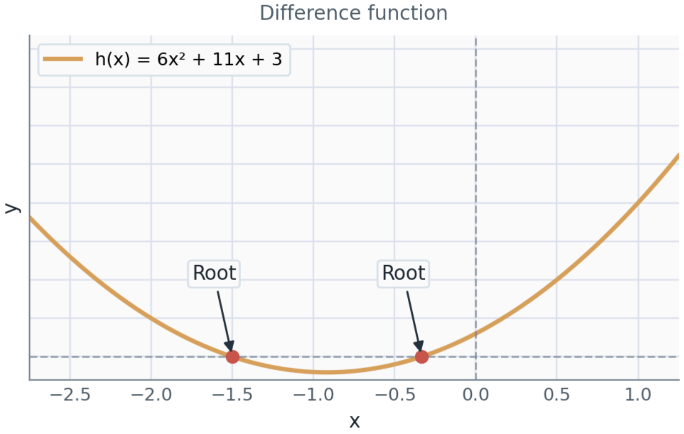
1. $4 = 2x$

2. $2x + 5x - 8 = 20$

Solving Quadratic Equations

- Example: Quadratic Equations as Roots





Consider functions:

- $f(x) = 8x^2 + 9x + 5$ (quadratic)
- $g(x) = 2x^2 - 2x + 2$ (quadratic)

Finding their intersection means determining x s.t:

$$f(x) = g(x) \quad \Leftrightarrow \quad 8x^2 + 9x + 5 = 2x^2 - 2x + 2$$

We can convert this to a root-finding problem by moving everything to the left-hand-side (LHS):

$$\begin{aligned} f(x) - g(x) &= 0 \\ \Leftrightarrow \quad (8x^2 + 9x + 5) - (2x^2 - 2x + 2) &= 0 \\ \Leftrightarrow \quad 8x^2 + 9x + 5 - 2x^2 + 2x - 2 &= 0 \\ \Leftrightarrow \quad 6x^2 + 11x + 3 &= 0 \end{aligned}$$

The LHS can be regarded as a new function $h(x)$.

Solving Quadratic Equations

- Example: Via Factorization

We factor the trinomial:

$$h(x) = 6x^2 + 11x + 3$$

Using factorization by grouping method:

1. Identify coefficients: $a = 6, b = 11, c = 3$.
2. Find integers m and n such that:
   - $mn = ac = 18$
   - $m + n = b = 11$

   Thus, choose $m = 9$ and $n = 2$.
3. Rewrite the middle term:

$$6x^2 + 11x + 3 = 6x^2 + \underbrace{9x}_{mx} + \underbrace{2x}_{nx} + 3$$

4. Group into pairs:

$$(6x^2 + 9x) + (2x + 3)$$

5. Factor the GCF from each group:

$$3x(2x + 3) + 1(2x + 3)$$

6. A Common binomial factor appears:

$$(2x + 3)(3x + 1)$$

7. We can now solve $h(x) = 0$, by solving each factor:

$$2x + 3 = 0 \ \Rightarrow \ x_1 = -\frac{3}{2},$$

$$3x + 1 = 0 \ \Rightarrow \ x_2 = -\frac{1}{3}.$$

Solving Quadratic Equations

- The Quadratic Formula

Consider the quadratic equation:

$$ax^2 + bx + c = 0, \quad a \neq 0$$

The solutions of this equation are given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $\Delta = b^2 - 4ac$ determines the number of real solutions:

- If $\Delta > 0$: two distinct real solutions.
- If $\Delta = 0$: one real (repeated) solution.
- If $\Delta < 0$: no real solutions.

Note: The $\pm$ symbol in the formula above indicates that we consider both the negative and positive square root.

Solving Quadratic Equations

- Example: Using the Quadratic Formula

To solve the quadratic equation

$$h(x) = 0 \quad \Leftrightarrow \quad 6x^2 + 11x + 3 = 0$$

We set $a = 6$, $b = 11$, and $c = 3$ in the formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-11 \pm \sqrt{11^2 - 4 \cdot 6 \cdot 3}}{2 \cdot 6} \\ &= \frac{-11 \pm \sqrt{121 - 72}}{12} \\ &= \frac{-11 \pm \sqrt{49}}{12} \end{aligned}$$

Hence, we get:

$$\begin{aligned} x_2 &= \frac{-11 + 7}{12} \\ &= -\frac{4}{12} = -\frac{1}{3} \end{aligned}$$

Or on the other hand:

$$\begin{aligned} x_1 &= \frac{-11 - 7}{12} \\ &= -\frac{18}{12} = -\frac{3}{2} \end{aligned}$$

These match the solutions obtained by factoring.

Verifying Solutions

- First Root: $x_1 = -3/2$

Whenever:

- We have solved an equation to obtain a solution
- We can always substitute the solution back into the equation to verify it

For example:

Consider the equation from earlier:

$$h(x) = 6x^2 + 11x + 3 = 0$$

With solution:

$$x_1 = -\frac{3}{2} \quad \text{and} \quad x_2 = -\frac{1}{3}$$

Substituting the solution x_1 into the equation we see:

$$6\left(-\frac{3}{2}\right)^2 + 11\left(-\frac{3}{2}\right) + 3 = 0$$
$$6\left(\frac{9}{4}\right) + 11\left(-\frac{3}{2}\right) + 3 = 0$$
$$\frac{54}{4} - \frac{33}{2} + 3 = 0$$
$$\frac{27}{2} - \frac{33}{2} + 3 = 0$$
$$-\frac{6}{2} + 3 = 0$$
$$-3 + 3 = 0 \quad \checkmark$$

Verifying Solutions

- Second Root: $x_2 = -1/3$

Whenever:

- We have solved an equation to obtain a solution
- We can always substitute the solution back into the equation to verify it

For example:

Consider the equation from earlier:

$$h(x) = 6x^2 + 11x + 3 = 0$$

With solution:

$$x_1 = -\frac{3}{2} \quad \text{and} \quad x_2 = -\frac{1}{3}$$

Substituting the solution x_2 into the equation we see:

$$\begin{aligned}
6\left(-\frac{1}{3}\right)^2 + 11\left(-\frac{1}{3}\right) + 3 &= 0 \\
6\left(\frac{1}{9}\right) + 11\left(-\frac{1}{3}\right) + 3 &= 0 \\
\frac{6}{9} - \frac{11}{3} + 3 &= 0 \\
\frac{2}{3} - \frac{11}{3} + 3 &= 0 \\
-\frac{9}{3} + 3 &= 0 \\
-3 + 3 &= 0 \quad \checkmark
\end{aligned}$$

Exercise Round 2

- Quadratic Equations

For the polynomial

$$f(x) = 2x^2 + 5x + 2,$$

1. Find its roots by factorization.

2. Find the same roots with the quadratic formula and interpret the discriminant.

3. Verify both roots by substitution.

Solving Inequalities

- Rules and Solution Sets

An inequality usually describes a set of values, i.e., not one isolated solution:

$$x > 2 \qquad \text{has solution set} \qquad (2, \infty).$$

State the answer as an inequality and as an interval.

When solving, the following operations preserve the direction:

- add or subtract the same quantity;
- multiply or divide by a positive number.

$$\begin{aligned} a < b \qquad\qquad &\implies \quad a + c < b + c, \\ a < b, \;\; c > 0 \qquad &\implies \quad ac < bc. \end{aligned}$$

There is one important exception: multiplying or dividing by a negative number reverses the inequality sign:

$$a < b,\; c < 0 \quad \implies \quad ac > bc.$$

Some inequalities combine several conditions. For these compound inequalities, apply each operation to all three parts:

$$\begin{aligned} 1 < 2x + 3 \le 7 \quad &\iff \quad -2 < 2x \le 4 \\ &\iff \quad -1 < x \le 2. \end{aligned}$$

Finally, check boundary points and a test value from each resulting interval.

Solving Linear Inequalities

- Example: Reversing the Inequality Sign

Solve

$$3 - 2x < 9.$$

Subtract 3 from both sides:

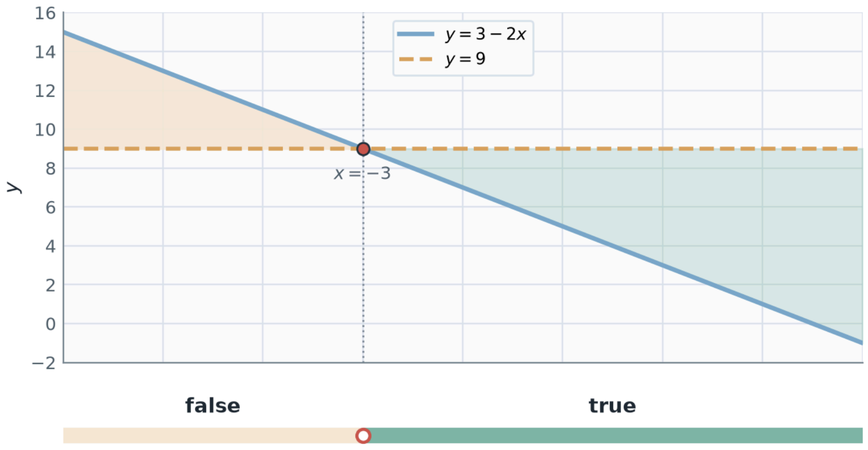
$$-2x < 6.$$

Divide by -2. Because it is negative, reverse the inequality sign:

$$x > -3.$$

Test one value on each side:

- $x = 0$: $3 - 2(0) = 3 < 9$ ✓
- $x = -4$: $3 - 2(-4) = 11 < 9$ is false



The graph of $y = 3 - 2x$ lies below $y = 9$ for $x > -3$. The boundary point is excluded because the inequality uses $<$, so

$$x > -3 \qquad \Longleftrightarrow \qquad x \in (-3, \infty).$$

Solving Quadratic Inequalities

- Example: Sign Analysis

Solve

$$x^2 - x - 6 \leq 0.$$

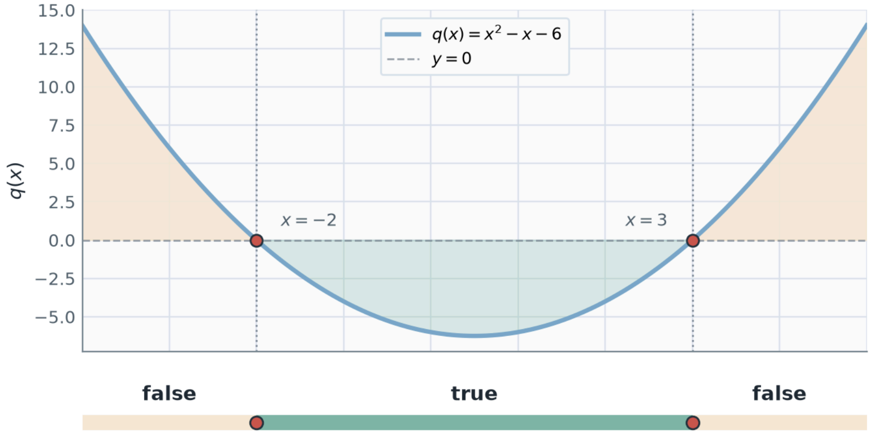
Factor the polynomial:

$$(x-3)(x+2) \leq 0.$$

The boundary points are $x = -2$ and $x = 3$. They split the real line into three intervals.

Test one value in each interval:

$$\begin{aligned} x = -3 &: \ (-6)(-1) = 6 > 0, \\ x = 0 &: \ (-3)(2) = -6 < 0, \\ x = 4 &: \ (1)(6) = 6 > 0. \end{aligned}$$



The graph is on or below the x-axis from -2 through 3. Both roots are included because the inequality uses $\leq$, so

$$-2 \leq x \leq 3 \quad \iff \quad x \in [-2, 3].$$

Exercise Round 3

- Inequalities

Solve each inequality over $\mathbb{R}$:

- give the answer in interval notation
- verify it using a boundary point or other suitable test values.

1. $5x - 7 \leq 13$
2. $4 - 3x > 10$
3. $-2 < 3x + 1 \leq 10$
4. $x^2 + x - 6 \geq 0$