Mathematics Brush-Up for Data Science

Lecture Slides 5: Functions II: Operations and Composition

Nicklas S. Andersen

University of Southern Denmark (SDU)

Department of Mathematics & Computer Science (IMADA)

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Combining Functions

- From Existing Functions to New Ones

So far, we have studied the characteristics of individual functions.

We can build new functions from existing ones in three closely related ways:

- combine their outputs using arithmetic operations,
- compose them so one output becomes another function's input, and
- decompose a complicated function into simpler inner and outer functions.

Together, these techniques allow us to:

- build models that combine several effects or stages
- break complex functions into simpler parts, making their domain restrictions easier to analyze.

Combining Functions

- Arithmetic Operations

Suppose $f : D_f \to \mathbb{R}$ and $g : D_g \to \mathbb{R}$ are real-valued functions.

Operation	Notation	Definition
Sum	$(f+g)(x)$	$f(x)+g(x)$
Difference	$(f-g)(x)$	$f(x)-g(x)$
Product	$(f \cdot g)(x)$	$f(x) \cdot g(x)$
Quotient	$\left(\frac{f}{g}\right)(x)$	$\frac{f(x)}{g(x)}$, provided $g(x) \neq 0$

- The sum, difference, and product have domain $D_f \cap D_g$.
- The quotient has domain $\{x \in D_f \cap D_g \mid g(x) \neq 0\}$.

Combining Functions

- Example: Subtraction and Division

Let

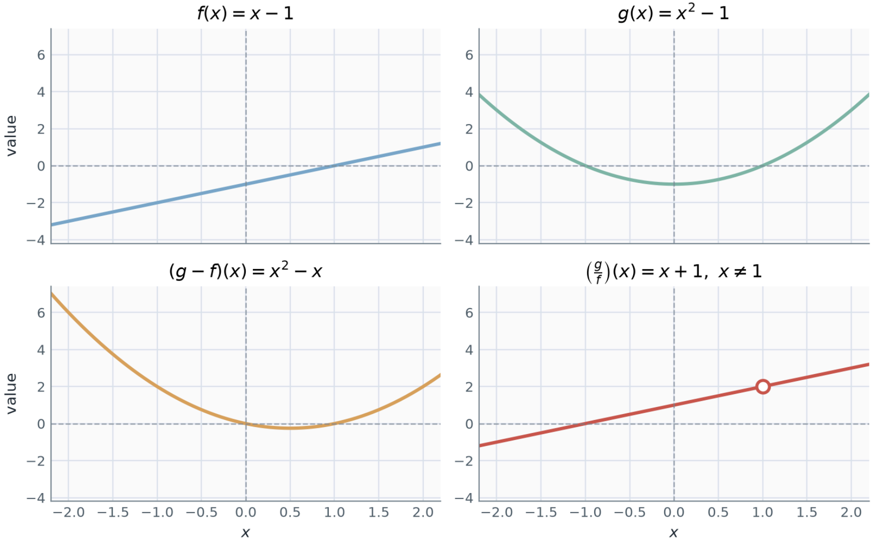
$$f(x) = x - 1, \qquad g(x) = x^2 - 1.$$

Subtract f from g:

$$\begin{aligned}(g - f)(x) &= (x^2 - 1) - (x - 1)\\ &= x^2 - x = x(x-1).\end{aligned}$$

Divide g by f:

$$\left(\frac{g}{f}\right)(x) = \frac{(x+1)(x-1)}{x-1} = x + 1, \quad x \neq 1.$$



The quotient has a hole at $x = 1$, even after its formula is simplified.

Combining Functions

- Example: Multiplication and Difference

Using the same functions,

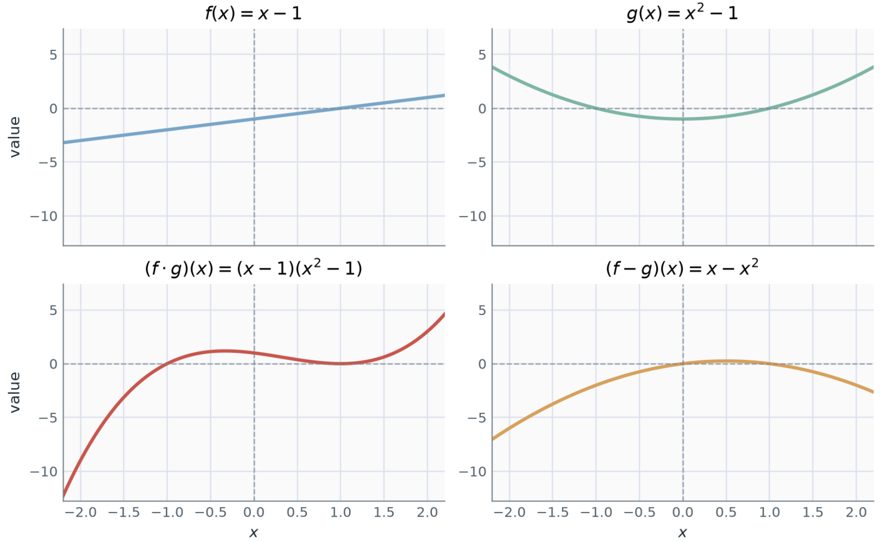
$$f(x) = x - 1, \qquad g(x) = x^2 - 1,$$

their product is

$$\begin{aligned}(f \cdot g)(x) &= (x-1)(x^2-1) \\ &= x^3 - x^2 - x + 1,\end{aligned}$$

while their difference is

$$(f - g)(x) = (x - 1) - (x^2 - 1) = x - x^2.$$



Different operations can produce functions of different classes and shapes.

Combining Functions

- Example: Models Built from Several Effects

Loss function

A model can balance prediction error $f(x)$ and a regularization penalty $g(x)$:

$$L(x) = f(x) + \lambda g(x), \qquad \lambda > 0.$$

The parameter λ controls how strongly model complexity is penalized.

Heat index

If $f(x)$ is temperature and $g(x)$ is humidity, a simple illustrative model is

$$h(x) = f(x) + 0.1g(x).$$

Both examples combine simpler functions into one quantity that captures several effects.

Function Composition

- A Chain of Dependencies

Let $g : A \to B$ and $f : B \to C$.

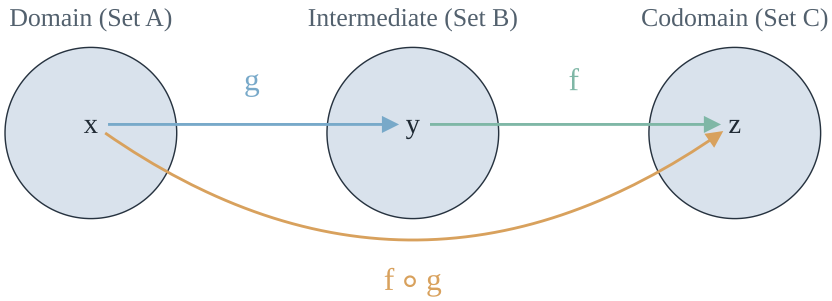
Composition describes a multi-step relationship:

1. apply g to the input x,
2. use the output $g(x)$ as the input to f.

The resulting function is

$$f \circ g : A \to C,$$
$$(f \circ g)(x) = f(g(x)).$$

Read from right to left: apply g first, then f.



Since $g(x) \in B$, every output of g is a valid input to f.

Thus, the domain of the composition is A, the domain of g:

$$D_{f \circ g} = A.$$

Function Composition

- Example: Electricity Use as a Composition

Suppose

- $T(d)$ gives the outdoor temperature on day d, and
- $E(T)$ gives the electricity needed for cooling at temperature T.

For a given day, temperature is evaluated first:

$$d \longmapsto T(d).$$

The resulting temperature is then used to evaluate electricity use:

$$d \longmapsto T(d) \longmapsto E(T(d)).$$

Thus, $(E \circ T)(10) = E(T(10))$ is the electricity used on day 10.

Function Composition

- Example: Order Matters

Let

$$f(x) = 2x + 1, \qquad g(x) = 3 - x.$$

Apply g first and then f:

$$(f \circ g)(x) = f(3 - x) = 2(3 - x) + 1 = 7 - 2x.$$

Reverse the order:

$$(g \circ f)(x) = g(2x + 1) = 3 - (2x + 1) = 2 - 2x.$$

Therefore, $f \circ g \neq g \circ f$. Composition is generally not commutative.

Function Composition

- Common Misconceptions

Composition is not multiplication

$$(f \circ g)(x) = f(g(x))$$

but

$$(f \cdot g)(x) = f(x)g(x).$$

One operation substitutes an output into another function; the other multiplies two outputs.

Composition may be undefined

The formula $f(g(x))$ only makes sense when $g(x)$ lies in the domain of f.

Changing the order can therefore change both:

- the formula of the result, and
- the set of valid inputs.

Exercise Round 1

- Operations and Composition

In the following let:

$$
\begin{aligned}
f(x) &= x - 1, \quad & g(x) &= x^2 - 1, \\
p(x) &= 2x + 1, \quad & q(x) &= x^2 - 4.
\end{aligned}
$$

Find and simplify each expression:

1. $(f + g)(x)$

2. $(f \cdot g)(x)$

3. $\left(\frac{g}{f}\right)(x)$

4. $(p \circ q)(x)$

5. $(q \circ p)(x)$

6. Explain why the two compositions are different.

Function Decomposition

- Reversing the Viewpoint

Decomposition expresses a complicated function as a composition of simpler functions:

$$h(x) = (f \circ g)(x) = f(g(x)).$$

1. Identify the expression evaluated first; this becomes the inner function g.
2. Identify the operation applied to that expression; this becomes the outer function f.
3. Recompose to verify that $f(g(x)) = h(x)$.

A function can have several valid decompositions; choose one that makes the next task easier.

Decomposing a function into simpler parts allows us to:

- identify the intermediate quantities involved,
- track domain restrictions at each stage, and
- among others, prepare nested functions for differentiation.

Function Decomposition

- Example: A Square-Root Function

Decompose

$$h(x) = \sqrt{5 - x^2}.$$

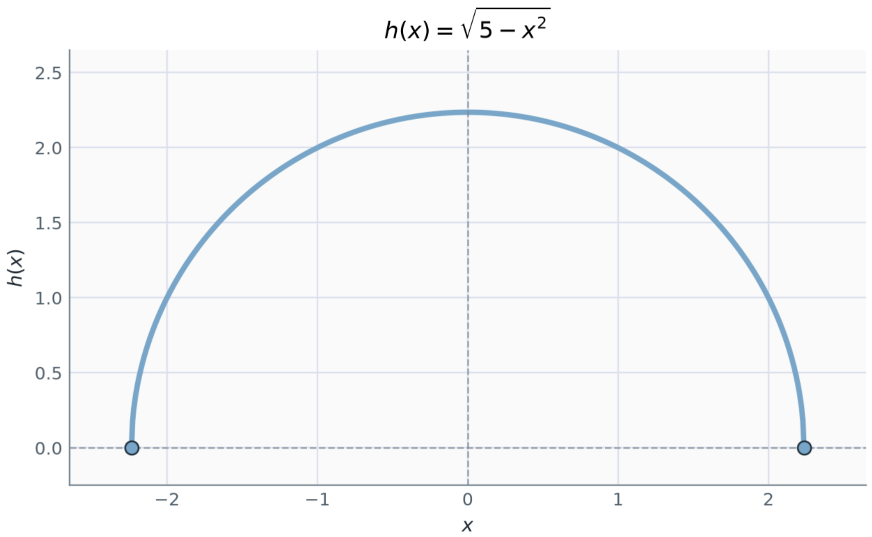
The expression $5 - x^2$ is evaluated before taking the square root, so choose

$$g(x) = 5 - x^2, \qquad f(x) = \sqrt{x}.$$

Check by recomposing:

$$f(g(x)) = f(5 - x^2) = \sqrt{5 - x^2} = h(x).$$

The domain is $[-\sqrt{5}, \sqrt{5}]$, where $g(x) \geq 0$ and the input to f is valid.



Function Decomposition

- Example: A Nested Expression

Decompose

$$h(x) = \frac{4}{3 - \sqrt{4 + x^2}}.$$

One useful choice is

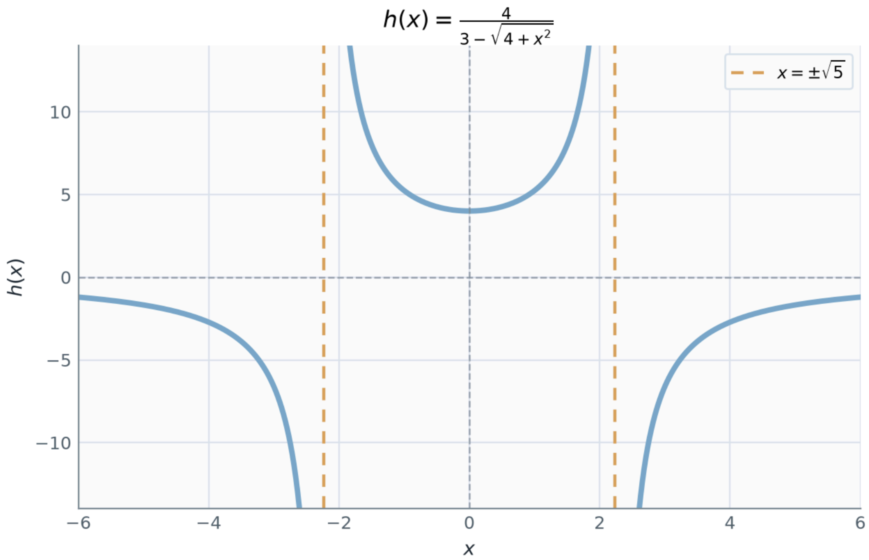
$$g(x) = \sqrt{4 + x^2}, \qquad f(x) = \frac{4}{3 - x}.$$

Then

$$f(g(x)) = \frac{4}{3 - \sqrt{4 + x^2}} = h(x).$$

The denominator is zero at $x = \pm\sqrt{5}$, so

$$D_h = \mathbb{R} \setminus \{-\sqrt{5}, \sqrt{5}\}.$$



Exercise Round 2

- Decomposition

For each function:

- choose an inner function g and an outer function f such that $h = f \circ g$;
- verify the decomposition by recomposing $f(g(x))$;
- state any restrictions needed over the real numbers.

1. $h(x) = \sqrt{3x + 5}$
2. $h(x) = \dfrac{1}{(x^2 + 1)^3}$
3. $h(x) = e^{x^2 - 4x}$
4. $h(x) = \ln(2x + 7)$