Mathematics Brush-Up for Data Science

Lecture Slides 4: Functions I: Fundamentals and Types

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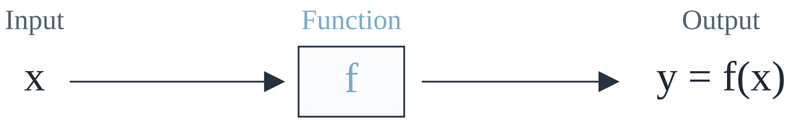
Functions

- Intuition & Terminology



A function (f) can be viewed as an input/output rule:

- for each valid input x
- it assigns exactly one output $y = f(x)$.

We often call:

- The input x the independent variable.
- The output y the dependent variable, because its value depends on x.

Functions

- Definition

A function f is a relation between two sets:

- A (the domain of f)
- B (the codomain of f)

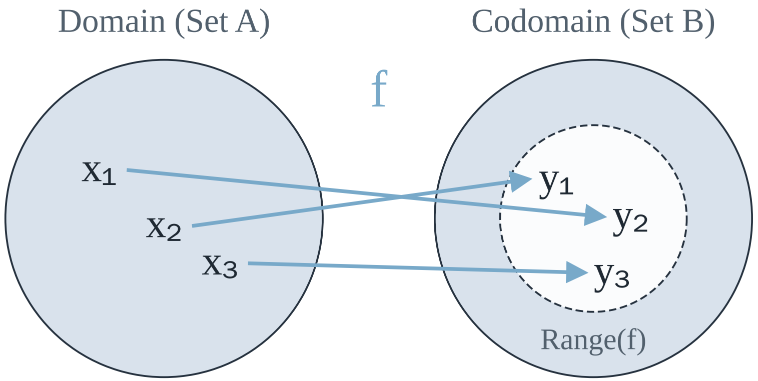
We write this as:

$$f : A \to B$$

- **Domain A:** all valid inputs.
- **Codomain B:** the declared set of possible outputs.
- **Range (image):** the outputs actually attained:

$$\text{Range}(f) = \{f(x) \mid x \in A\} \subseteq B$$

If no domain is stated, use the real domain: all real input values for which the formula is defined and produces a real output.



Functions

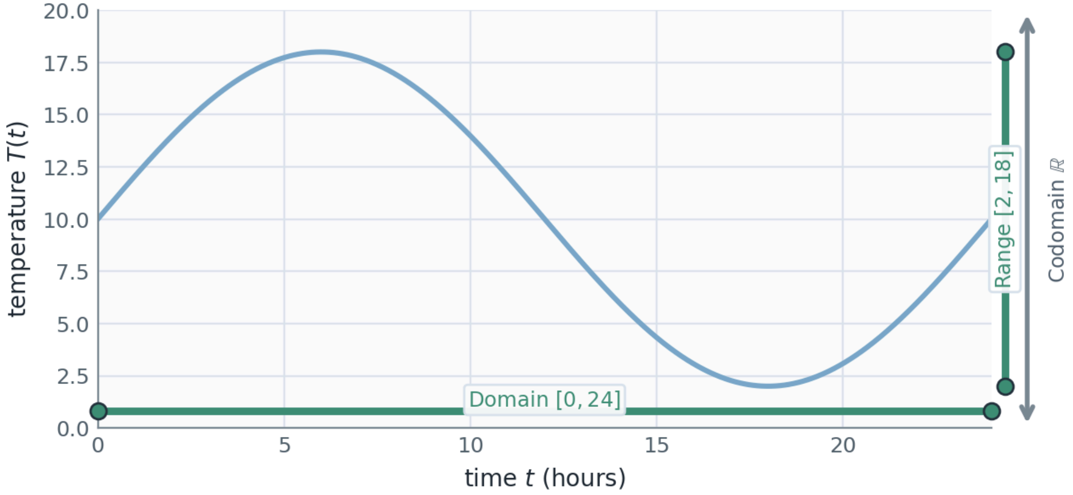
- Example: Temperature Over a Day

Suppose the outdoor temperature is tracked over one day. Let t denote the elapsed time in hours since midnight. The temperature, in $^\circ\mathrm{C}$, is modeled by

$$T(t) = 10 + 8\sin\left(\frac{\pi t}{12}\right).$$

- **Domain:** $[0, 24]$
- **Codomain:** $\mathbb{R}$
- **Range:** $[2, 18]$

The sine term varies from -1 to 1, so scaling by 8 and shifting by 10 gives the stated range.



Functions

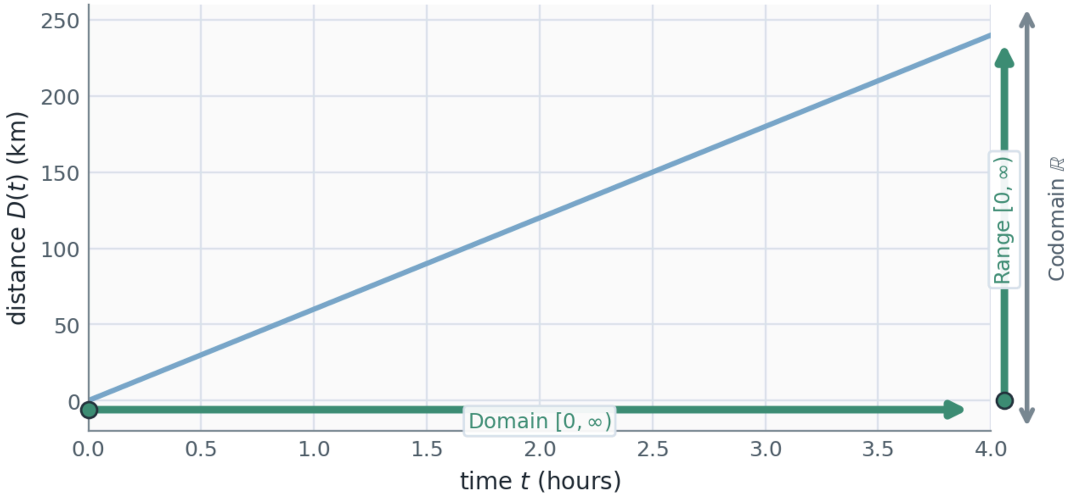
- Example: Distance at Constant Speed

Suppose a car travels at a constant speed of $60\,\mathrm{km}/\mathrm{h}$. Let t denote the elapsed travel time in hours. The distance traveled, in kilometers, is modeled by

$$D(t) = 60t.$$

- **Domain:** $[0, \infty)$
- **Codomain:** $\mathbb{R}$
- **Range:** $[0, \infty)$

Non-negative time produces non-negative distance, starting from $D(0) = 0$.



Functions

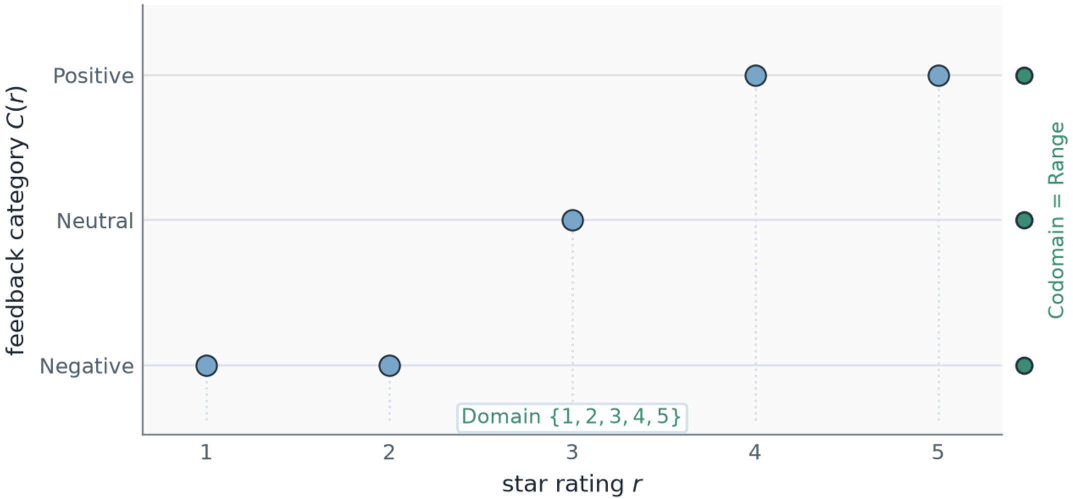
- Example: Customer Feedback from a Star Rating

Suppose a platform groups customer star ratings into feedback categories. Let r denote the number of stars. The category $C(r)$ is defined by:

$$\begin{aligned} C(1) = C(2) &= \text{Negative}, \\ C(3) &= \text{Neutral}, \\ C(4) = C(5) &= \text{Positive}. \end{aligned}$$

- **Domain:** $\{1, 2, 3, 4, 5\}$
- **Codomain:** $\{\text{Negative}, \text{Neutral}, \text{Positive}\}$
- **Range:** $\{\text{Negative}, \text{Neutral}, \text{Positive}\}$

Both the inputs and outputs form discrete sets.



Exercise Round 1

- Domain, Codomain & Range

For each situation, suggest an appropriate domain, codomain, and range. State any assumptions.

1. The battery percentage $B(t)$ of a phone, where t is the time in hours since full charge.

2. An exam has 20 one-point questions, where s is the number of correct answers:

$$O(s) = \begin{cases} \text{Fail}, & s < 10, \\ \text{Pass}, & s \geq 10. \end{cases}$$

3. The relative humidity $H(t)$, where t is the hour of the day.

Functions

- Representation Methods

Functions can be represented in different ways:

- Each representation highlights different insights
- Choice depends on context and purpose

To illustrate these representations, we will use the simple example from earlier:

$$f(x) = \text{ crop yield for fertilizer amount } x$$

That is, $f(x)$ denotes the crop yield (t/ha) as a function of fertilizer amount x (kg/ha).

Representation Methods

- Tables

Fertilizer x $(\mathrm{kg/ha})$	Crop yield $f(x)$ $(\mathrm{t/ha})$
0	3.4942
1	3.5038
2	3.5133
$\vdots$	$\vdots$
197	4.1589
200	4.1500
$\vdots$	$\vdots$
400	2.2694

A table lists selected input/output pairs and gives exact values at the listed inputs.

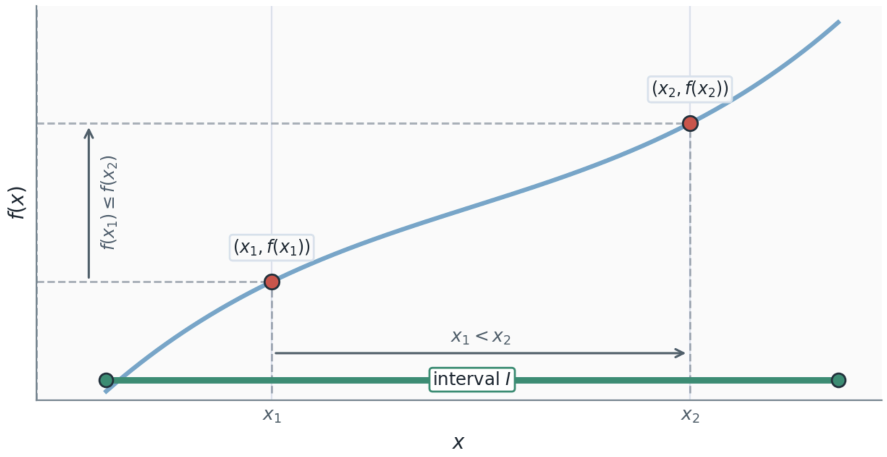
For example,

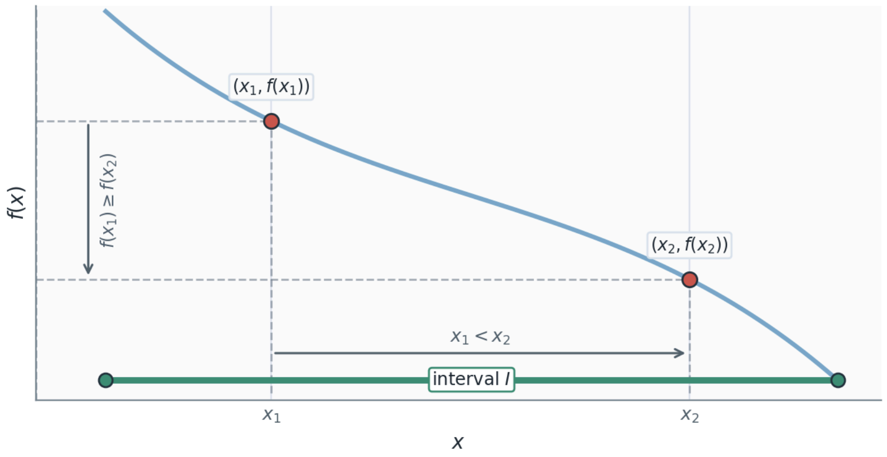
$$f(200) = 4.1500.$$

The values suggest that yield first rises and later falls, but a table alone does not show the overall shape clearly.

Function Behavior

- Increasing and Decreasing on an Interval





Increasing on I: if $x_1 < x_2$, then

$$f(x_1) \leq f(x_2).$$

For a strictly increasing function, replace $\leq$ with $<$.

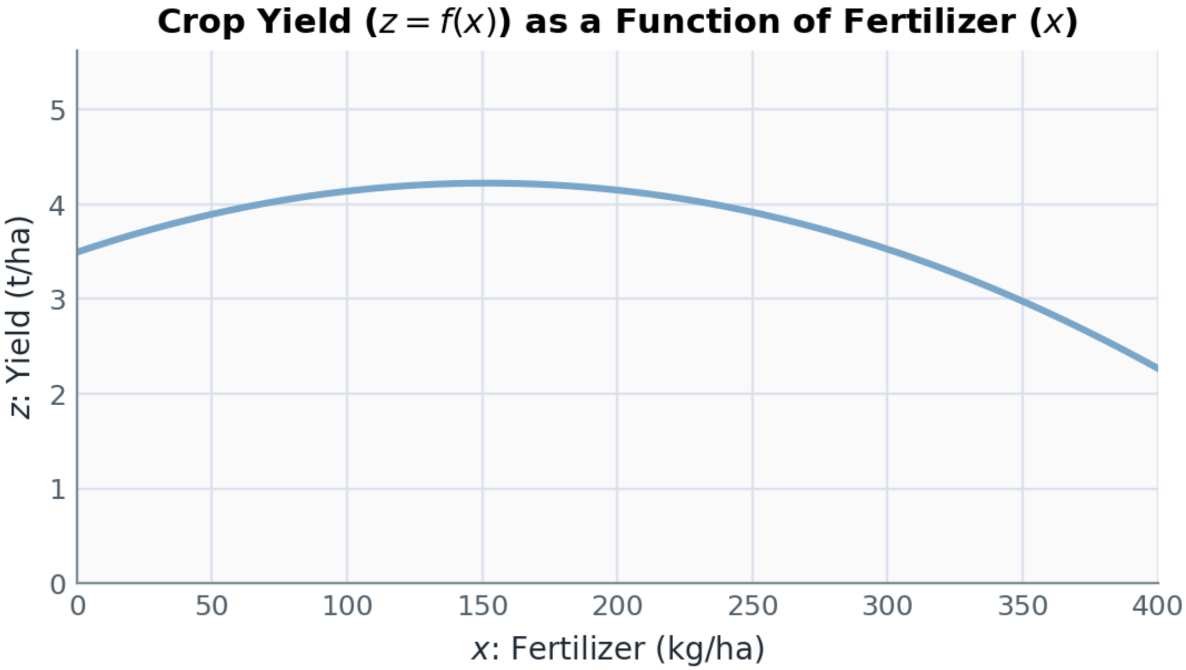
Decreasing on I: if $x_1 < x_2$, then

$$f(x_1) \geq f(x_2).$$

For a strictly decreasing function, replace $\geq$ with $>$.

Representation Methods

- Graphs



If A is the domain of f, its graph is the set of points

$$(x, f(x)), \qquad x \in A,$$

plotted in a coordinate plane.

Here the graph reveals that yield rises, flattens near a peak, and then falls. The relationship is nonlinear.

Representation Methods

- Algebraic Formulas

An algebraic formula provides:

- A general rule for any valid input
- Compact, symbolic way to describe relationships

The table and graph were synthetically generated from the function:

$$f(x) = -3.17042 \cdot 10^{-5} \cdot x^2 + 0.00961968x + 3.49418.$$

This formula is not arbitrary: it was fitted to the synthetic data to approximate their overall pattern.

Benefits of an algebraic formula:

Interpolation
We can estimate outputs between known input values.

Extrapolation
We can estimate beyond the observed range, though with greater uncertainty.

Equation solving
We can solve $f(x) = 4.0$ to find fertilizer amounts x predicting $4.0\,\mathrm{t/ha}$.

Basic Classes of Functions

- Common Graphical Features

Functions can be grouped into different classes based on their algebraic form.

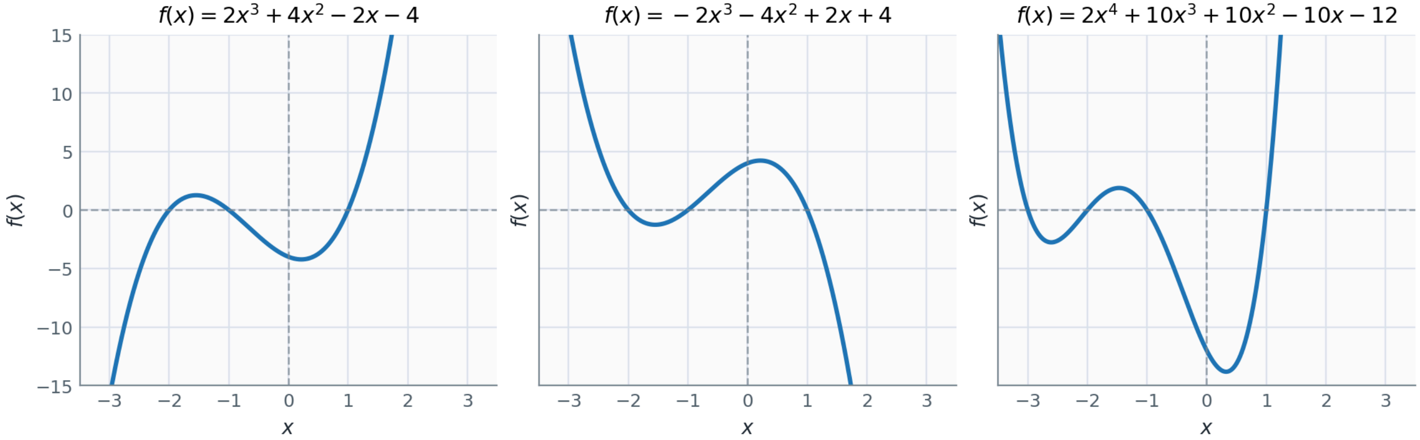
Each class has its own properties, domain and range, and characteristic graph shape.

Some common and important features are:

Feature	Definition
Intercepts	Points where the graph meets the coordinate axes.
Turning Points	Points where the graph changes direction from increasing to decreasing, or vice versa.
Asymptotes	Lines describing limiting behavior near a boundary or as the input grows without bound. Horizontal and oblique asymptotes may be crossed.

Polynomial Functions

- Definition



Every nonzero polynomial can be written in the general form:

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

Here:

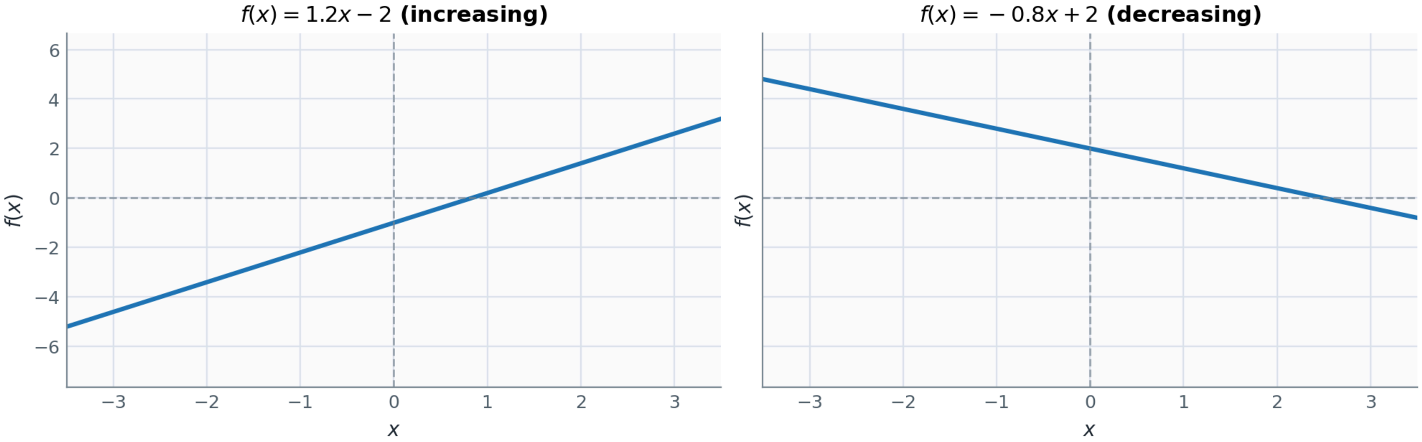
- Degree: $n = 0, 1, 2, \ldots$
- Coefficients: $a_n, a_{n-1}, \ldots, a_0$ are real constants
- Leading coefficient: $a_n \neq 0$

A special case:

- The zero polynomial $f(x) = 0$ is also a polynomial, but its degree is left undefined in this course.

Polynomial Functions (Degree 1)

- Domain, Range, and Graph



A linear function is a polynomial of degree 1; its graph is a straight line and can be written in the general form:

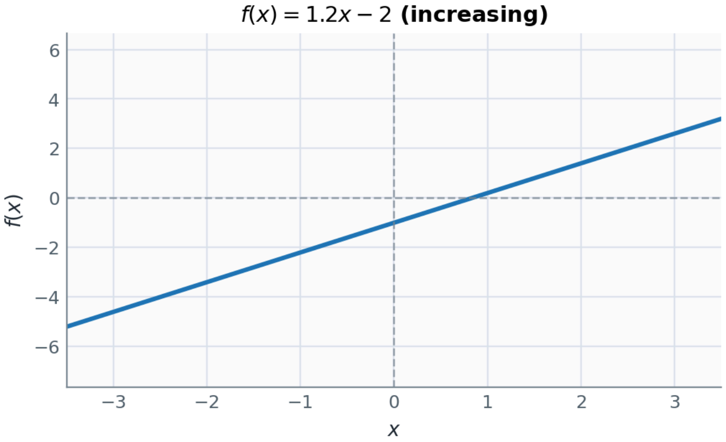
$$f(x) = ax + b$$

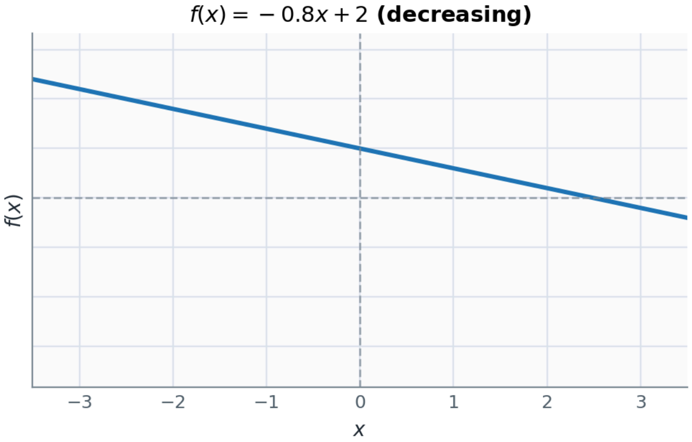
Here:

- a and b are real constants
- $a \neq 0$

Polynomial Functions (Degree 1)

- Definition





The general form of a linear function:

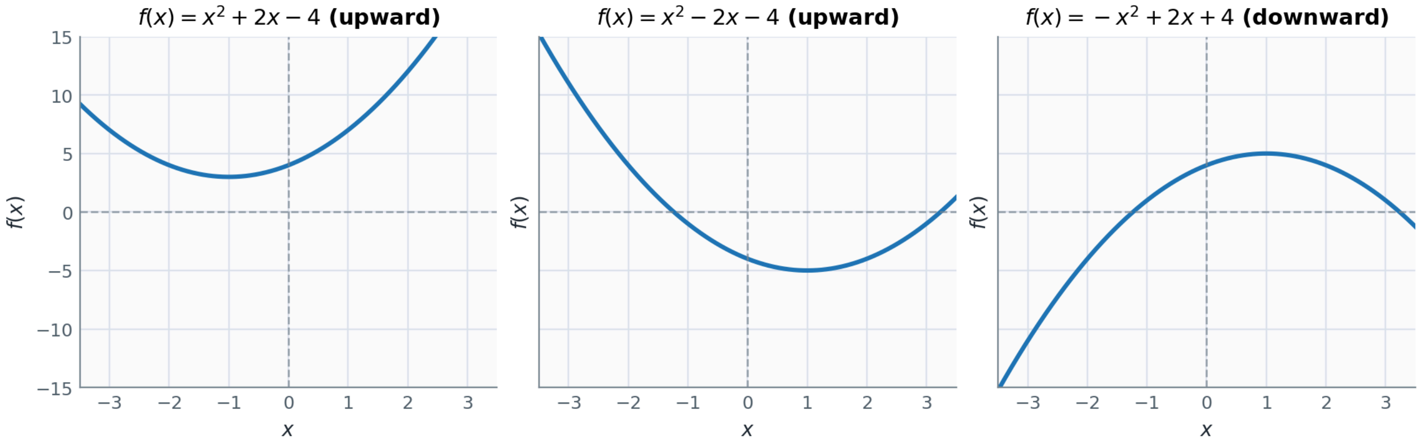
$$f(x) = ax + b$$

- Domain: All real numbers ($\mathbb{R}$)
- Range: All real numbers ($\mathbb{R}$)

- Graph: A straight line with slope a:
  - If $a > 0$ the function is increasing
  - If $a < 0$ the function is decreasing
- Intercepts:
  - y-intercept at $(0, b)$
  - x-intercept at $(\frac{-b}{a}, 0)$

Polynomial Functions (Degree 2)

- Domain, Range, and Graph



A quadratic function is a polynomial of degree 2; its graph is a parabola and can be written in the general form:

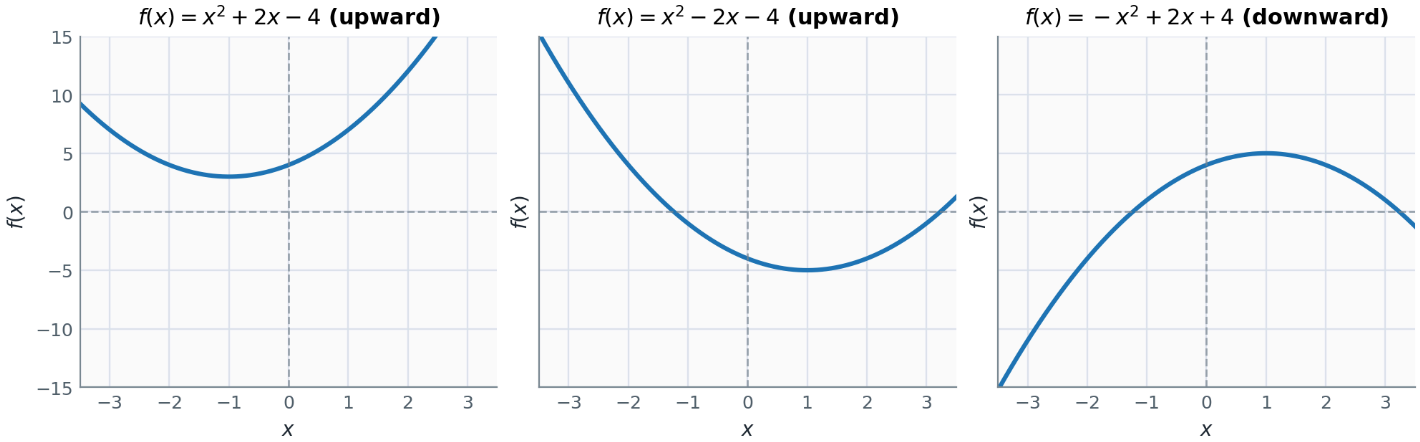
$$f(x) = ax^2 + bx + c$$

Here:

- a, b, and c are real constants
- $a \neq 0$

Polynomial Functions (Degree 2)

- Definition



The general form of a quadratic function:

$$f(x) = ax^2 + bx + c$$

- Domain: All real numbers ($\mathbb{R}$)
- Range:
  - If $a > 0$ it is $[y_{\min}, \infty)$
  - If $a < 0$ it is $(-\infty, y_{\max}]$

- Graph: Opening up ($a > 0$) or down ($a < 0$)
- Intercepts: Max 2 x-intercepts, exactly 1 y-intercept
- Turning Point:
  - If $a > 0$, it is the lowest point (a minimum)
  - If $a < 0$, it is the highest point (a maximum)

Polynomial Functions

- Terminology

Characterizing polynomials based on **the degree**:

Degree	Name	Example
0	Constant	7
1	Linear	$2x+3$
2	Quadratic	x^2-4x+4
3	Cubic	x^3-x
4	Quartic	x^4+2x^2+1
5	Quintic	x^5-x^3+1
$n \geq 6$	nth-degree polynomial	$x^6+\dots$

Polynomial Functions

- Terminology (Continued)

Characterizing polynomials based on **the number of terms**:

Number of Terms	Name	Example
1	Monomial	$5x^3$
2	Binomial	$3x^2 + 1$
3	Trinomial	$x^2 - 4x + 4$
≥ 4	Polynomial	$x^4 + x^3 - 2x + 7$

Exercise Round 2

- Polynomial Functions

Consider the functions

$$
\begin{aligned}
f(x) &= 3x - 5, \qquad & g(x) &= -2x^2 + 8, \\
h(x) &= x(x-4), & p(x) &= 7, \\
q(x) &= 2^x.
\end{aligned}
$$

For each function, complete the following tasks.

1. Decide whether it is a polynomial.

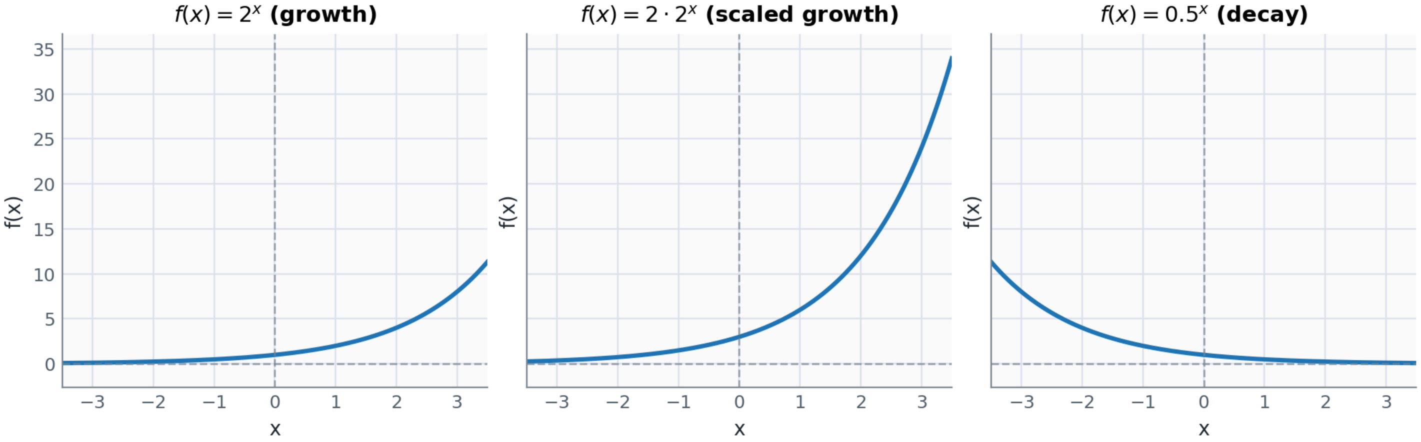
2. For each polynomial, state:

- its degree and degree-based name
- whether it is a monomial, binomial, or trinomial.

3. Describe the relevant graph behavior.

Exponential Functions

- Domain, Range, and Graph



Exponential functions have a constant base raised to a variable exponent and can be written in the general form:

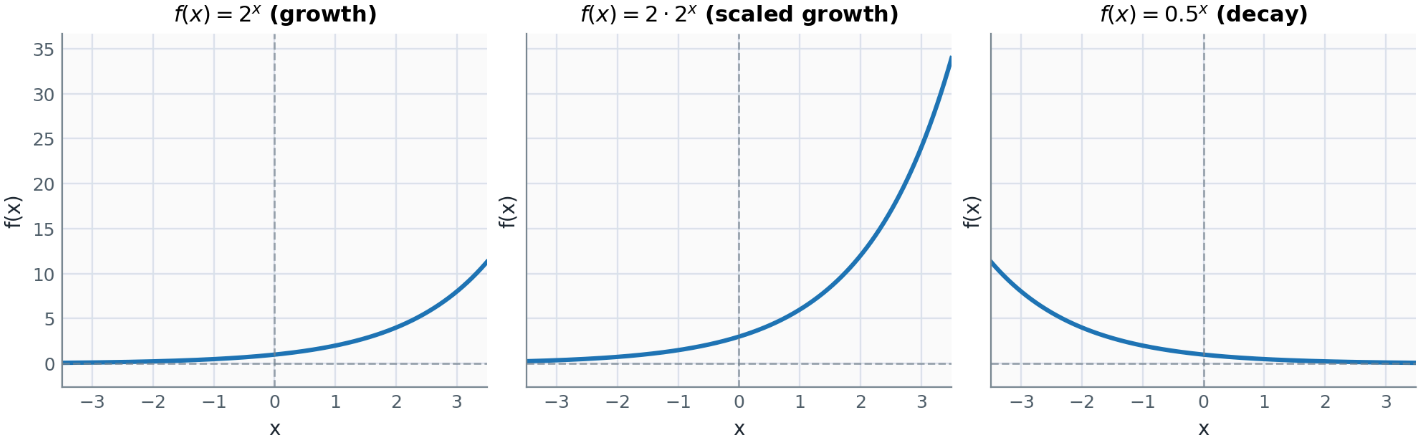
$$f(x) = ab^x$$

Here:

- $a \neq 0$
- $b > 0$, and $b \neq 1$

Exponential Functions

- Definition



The general form of an exponential function:

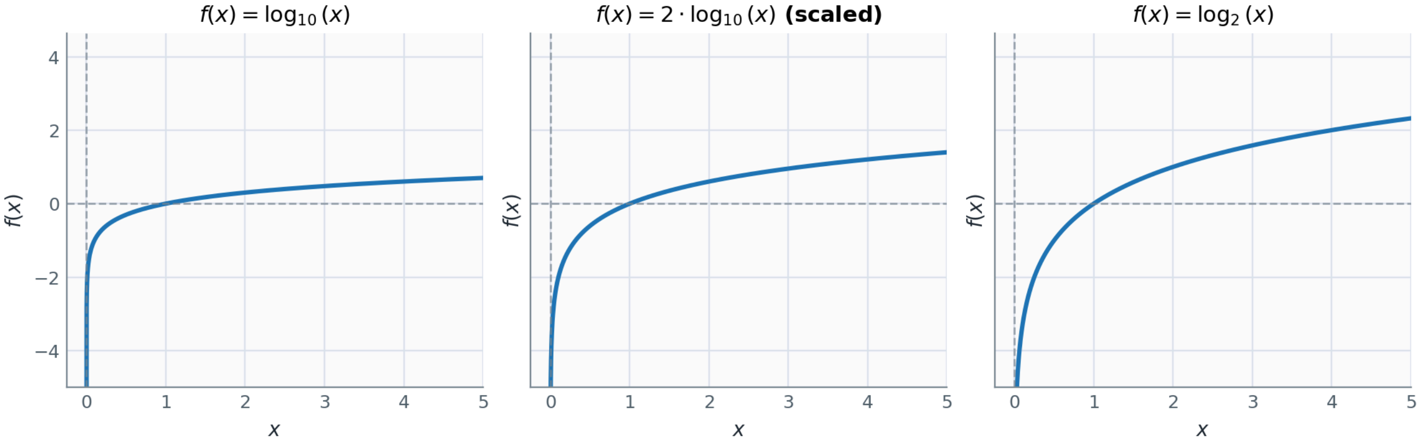
$$f(x) = ab^x$$

- Domain: All real numbers ($\mathbb{R}$)
- Range:
  - If $a > 0$ it is $(0, \infty)$
  - If $a < 0$ it is $(-\infty, 0)$

- Graph:
  - If $a > 0$ and $b > 1$, the graph shows growth
  - If $a > 0$ and $0 < b < 1$, the graph shows decay
  - If $a < 0$, the graph is reflected across the x-axis and these directions are reversed
- Scaling: $|a|$ scales the graph vertically
- Asymptote: Horizontal at $y = 0$

Logarithmic Functions

- Domain, Range, and Graph



The basic logarithmic function $\log_b(x)$ is the inverse of the exponential function b^x. A broader family of logarithmic functions can be written as:

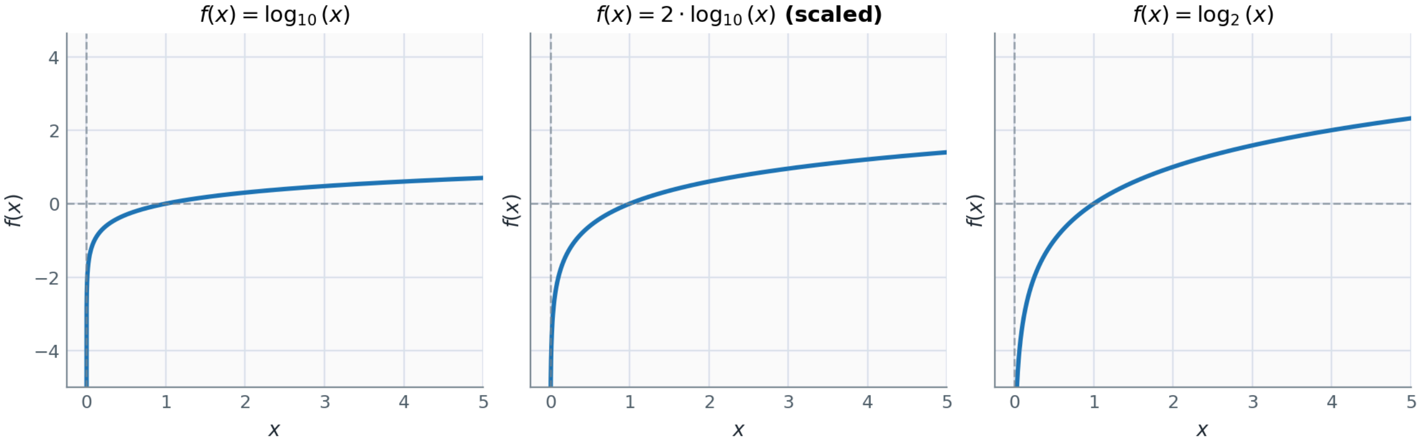
$$f(x) = a \cdot \log_b(x)$$

Here:

- $a \neq 0$
- $b > 0$, and $b \neq 1$

Logarithmic Functions

- Definition



The general form of a logarithmic function:

$$f(x) = a \cdot \log_b(x)$$

- Domain: $(0, \infty)$
- Range: $\mathbb{R}$

- Graph:
  - The graph passes through $(1, 0)$
  - If $a > 0$ and $b > 1$, the graph is increasing
  - If $a > 0$ and $0 < b < 1$, the graph is decreasing
  - If $a < 0$, the graph is reflected across the x-axis and these directions are reversed
- Scaling: $|a|$ scales the graph vertically
- Asymptote: Vertical at $x = 0$

Piecewise Functions

- Definition

Let $f : A \to B$. A piecewise-defined function uses a separate expression $f_i(x)$ on each subset $A_i \subseteq A$:

$$f(x) = \begin{cases} f_1(x), & x \in A_1, \\ f_2(x), & x \in A_2, \\ \vdots & \vdots \\ f_n(x), & x \in A_n. \end{cases}$$

- Each rule f_i applies on a subset A_i.
- The subsets $A_1, \ldots, A_n$ are pairwise disjoint.
- Together, they form the entire domain.
- Each input uses exactly one rule and has one output.
- Continuity may hold or fail at a rule change.

Domain: Combine all rule-specific input subsets.

$$A = A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$

Range: Combine all outputs produced by the rules.

$$\mathrm{Range}(f) = \bigcup_{i=1}^{n} \{f_i(x) \mid x \in A_i\} \subseteq B$$

Piecewise Functions

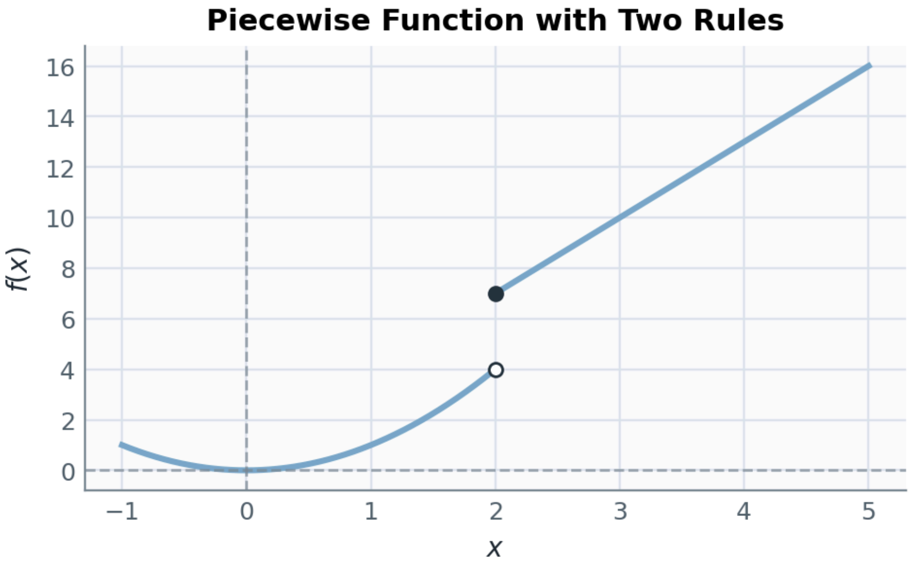
- Example: Piecewise Function with Two Rules

A piecewise function might be defined as:

$$f(x) = \begin{cases} 3x+1, & \text{if } x \geq 2, \\ x^2, & \text{if } x < 2. \end{cases}$$

To evaluate f at a given input, first check whether the input is below 2 or at least 2. This determines which rule applies:

$$\begin{aligned} 5 \geq 2 \quad &\Rightarrow \quad f(5) = 3 \cdot 5 + 1 = 16, \\ -1 < 2 \quad &\Rightarrow \quad f(-1) = (-1)^2 = 1. \end{aligned}$$

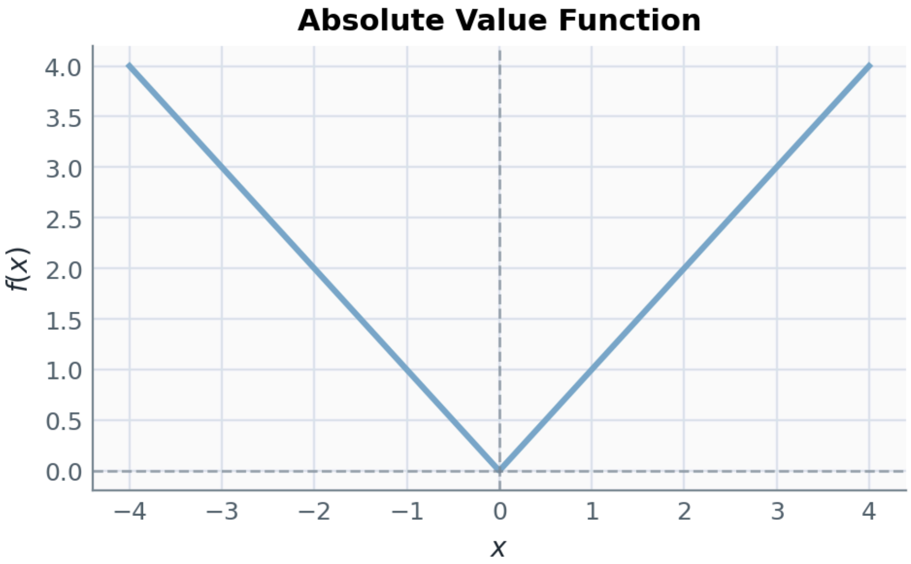


Absolute Value Function

- Example: A Familiar Piecewise-Defined Function

The absolute value function:

$$f(x) = \begin{cases} x, & \text{if } x \geq 0, \\ -x, & \text{if } x < 0. \end{cases}$$



Piecewise Functions

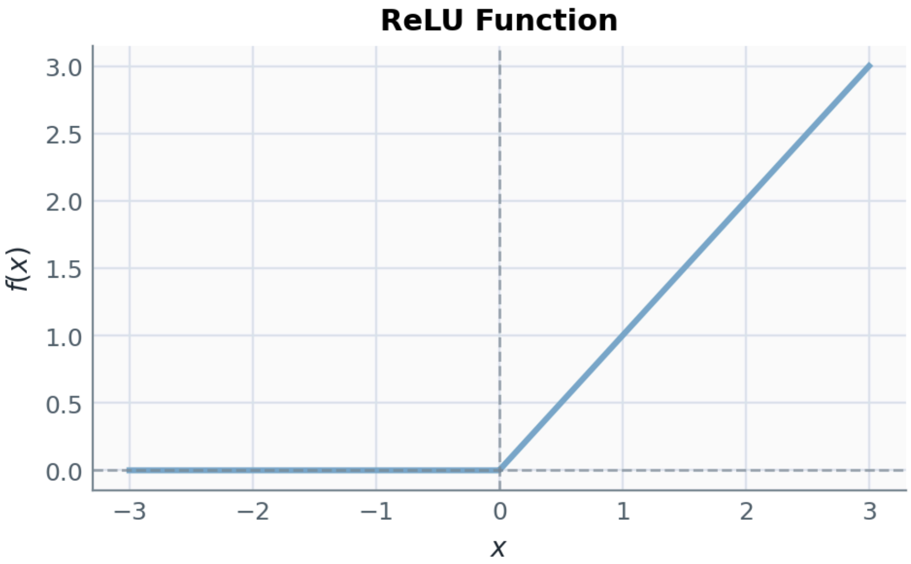
- Example: ReLU Function

The **Rectified Linear Unit (ReLU)** is a commonly used activation function in neural networks:

$$\mathrm{ReLU}(x) = \begin{cases} x, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

It clips negative inputs to zero while leaving positive inputs unchanged. Equivalently,

$$\mathrm{ReLU}(x) = \max(0, x).$$



Exercise Round 3

- Basic Function Classes

Classify each function by its basic function class.

1. $f(x) = 7$
2. $f(x) = \log_2(x)$
3. $f(x) = 3x - 5$
4. $f(x) = \ln(x)$
5. $f(x) = x^2$
6. $f(x) = 5^x$
7. $f(x) = 2x^2 + 1$
8. $f(x) = x(x - 5)$
9. $f(x) = 5 \cdot (0.5)^x$