Mathematics Brush-Up for Data Science

Lecture Slides 3: Basic Algebra

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Basic Algebra

Algebra is the branch of mathematics that:

- Uses symbols and letters to represent numbers (variables)
- Applies rules to manipulate these symbols

In the following, we will explore these ideas in the context of:

- Fractions
- Exponents

Fractions

A fraction is a rational number and can be written as:

$$\frac{p}{q} \in \mathbb{Q} \qquad \text{with} \qquad p, q \in \mathbb{Z}, \quad q \neq 0.$$

- p is the numerator: the number above the line
- q is the denominator: the nonzero number below the line

Fractions represent parts of a whole and are used to express:

- Proportions
- Ratios
- Percentages

The rational numbers form a proper subset of the real numbers: $\mathbb{Q} \subset \mathbb{R}$.

Fraction Rules

- Adding & Subtracting Fractions

When the denominators already match, add or subtract the numerators:

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}, \qquad c \neq 0$$

Adding numerators and denominators separately is not a valid rule.

For example, this approach is incorrect:

$$\frac{1}{3} + \frac{1}{3} \neq \frac{1+1}{3+3} = \frac{2}{6} = \frac{1}{3}$$

The correct calculation is:

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Fraction Rules

- Common Denominators

With different denominators, use cd as a common denominator:

$$\frac{a}{c} \pm \frac{b}{d} = \frac{ad \pm bc}{cd}, \qquad c, d \neq 0$$

For example, when adding:

$$\frac{1}{4} + \frac{2}{3} = \frac{1 \cdot 3 + 2 \cdot 4}{4 \cdot 3} = \frac{11}{12}$$

The same method applies when subtracting:

$$\frac{5}{6} - \frac{1}{3} = \frac{5 \cdot 3 - 1 \cdot 6}{6 \cdot 3} = \frac{9}{18} = \frac{1}{2}$$

Fraction Rules

- Multiplying Fractions

Multiplication of fractions is straightforward:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \qquad b, d \neq 0$$

We simply multiply the numerators and denominators.

For example:

$$\frac{1}{4} \cdot \frac{2}{3} = \frac{1 \cdot 2}{4 \cdot 3} = \frac{2}{12} = \frac{1}{6}$$

Any integer can be written as a fraction with denominator 1, so multiplication by an integer follows the same rule.

For example:

$$3 \cdot \frac{2}{5} = \frac{3}{1} \cdot \frac{2}{5} = \frac{3 \cdot 2}{1 \cdot 5} = \frac{6}{5}$$

Fraction Rules

- Dividing Fractions

To divide fractions, multiply the first fraction by the reciprocal of the second:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}, \qquad b, c, d \neq 0$$

For example:

$$\frac{\frac{1}{4}}{\frac{2}{3}} = \frac{1}{4} \cdot \frac{3}{2} = \frac{1 \cdot 3}{4 \cdot 2} = \frac{3}{8}$$

Exercise Round 1

- Fractions

1. Simplify:

$$\frac{2}{5} + \frac{7}{10}$$

2. Determine if the following is true or false:

$$\frac{2}{7} + \frac{3}{7} = \frac{5}{14}$$

3. Simplify:

$$\frac{3}{8} \cdot \frac{4}{9}$$

4. Evaluate the following expression:

$$\frac{\frac{7}{12}}{\frac{14}{9}}$$

5. Is the following simplification from left to right valid for all $x \neq 0$? Explain.

$$\frac{x+2}{x} = 1 + 2, \qquad \text{for } x \neq 0.$$

Exponents

- Products & Powers

Exponents indicate how many times a base number is multiplied by itself, e.g:

$$2^2 = 2 \cdot 2 \quad \text{or} \quad 4^3 = 4 \cdot 4 \cdot 4$$

The Product Rule: When multiplying terms with the same base, we add their exponents:

$$a^n \cdot a^m = a^{n+m}, \qquad a \in \mathbb{R} \setminus \{0\} \text{ and } n, m \in \mathbb{Z}$$

For example: $2^4 \cdot 2^2 \;=\; 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \;=\; 2^6$.

The Power Rule: When raising an exponential term to another power, we multiply the exponents:

$$(a^n)^m = a^{n \cdot m}, \qquad a \in \mathbb{R} \setminus \{0\} \text{ and } n, m \in \mathbb{Z}$$

For example: $(5^3)^2 \;=\; (5 \cdot 5 \cdot 5) \cdot (5 \cdot 5 \cdot 5) \;=\; 5^6$.

Exponents

- Negative Exponents & The Quotient Rule

Negative Exponents: A base to a negative exponent is equal to the reciprocal of the base to the positive exponent:

$$a^{-n} = \frac{1}{a^n}, \qquad a \in \mathbb{R} \setminus \{0\} \text{ and } n \in \mathbb{Z}_{>0}.$$

For example: $2^{-3} \ = \ \frac{1}{2^3} \ = \ \frac{1}{8}$

The Quotient Rule: When dividing terms with the same base, we subtract their exponents.

$$\begin{aligned} \frac{a^m}{a^n} &= a^m \cdot \frac{1}{a^n} \\ &= a^m \cdot a^{-n} \\ &= a^{m-n} \end{aligned} \qquad a \in \mathbb{R} \setminus \{0\} \text{ and } m, n \in \mathbb{Z}.$$

For example: $\frac{x^5}{x^2} \ = \ x^5 \cdot \frac{1}{x^2} \ = \ x^5 \cdot x^{-2} \ = \ x^{5-2} = x^3$, where $x \neq 0$.

Exercise Round 2

- Integer Exponents

1. Simplify:

$$(x^3)^4$$

2. Simplify:

$$\frac{5^6}{5^2}$$

3. Determine if the following is true or false:

$$x^{-2} = \frac{1}{x^2}, \qquad \text{for } x \neq 0.$$

4. Simplify:

$$\frac{y^{-3}}{y^{-7}}, \qquad \text{for } y \neq 0.$$

5. Simplify:

$$\frac{2^3}{4^{-2}} \cdot \frac{3}{\frac{9}{8}}$$

Exponents

- Roots as Exponents

Roots can be expressed as fractional exponents. For a positive integer r:

$$a^{\frac{1}{r}} = \sqrt[r]{a}, \qquad r \in \mathbb{Z}_{>0}, \qquad \begin{cases} a \geq 0, & r \text{ even}, \\ a \in \mathbb{R}, & r \text{ odd}. \end{cases}$$

For example:

- For $a \geq 0$, the principal square root satisfies $\sqrt{a} = a^{\frac{1}{2}}$.
- For $a \in \mathbb{R}$, the cube root satisfies $\sqrt[3]{a} = a^{\frac{1}{3}}$.

Exponents

- Edge Cases & Comments

Zero requires special care when it appears as either a base:

$$0^a = \begin{cases} 0, & a > 0, \\ \text{undefined}, & a \leq 0, \end{cases} \qquad a \in \mathbb{R}.$$

or an exponent:

$$a^0 = 1, \qquad a \in \mathbb{R} \setminus \{0\}.$$

Some comments:

- For positive bases:
  - Exponent rules extend from integer to real exponents.
  - Zero and negative bases require additional domain checks.
- The same-base product rule does not apply to $2^3 \cdot 3^5$ because the bases differ.
- Exponent rules do not combine terms in a sum such as $2^3 + 3^5$.

Exponents

- Products of Roots

The root of a product can be separated into a product of roots under the following:

$$(ab)^{\frac{1}{r}} = a^{\frac{1}{r}} b^{\frac{1}{r}}, \qquad r \in \mathbb{Z}_{>0}, \qquad \begin{cases} a, b \geq 0, & r \text{ even}, \\ a, b \in \mathbb{R}, & r \text{ odd}. \end{cases}$$

For example:

$$\sqrt{12} = 12^{\frac{1}{2}} = (4 \cdot 3)^{\frac{1}{2}} = 4^{\frac{1}{2}} 3^{\frac{1}{2}} = \sqrt{4}\sqrt{3} = 2\sqrt{3}$$

Exercise Round 3

- Roots & Fractional Exponents

1. Express the following as a root and evaluate:

$$27^{\frac{2}{3}}$$

2. Use the product-of-roots rule to simplify:

$$\sqrt{72}$$

3. Evaluate the following expression:

$$a^{-2} \cdot b^{\frac{3}{2}}$$

given that

$$a = \frac{1}{2} \qquad \text{and} \qquad b = 4^{-1}.$$