Mathematics Brush-Up for Data Science

Lecture Slides 14: Mathematical Logic

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Mathematical Logic

Mathematical logic is the branch of mathematics that:

- Studies reasoning using precise rules and symbols
- Represents, combines, and compares statements according to these rules

This gives us a clear and systematic way to describe and examine reasoning.

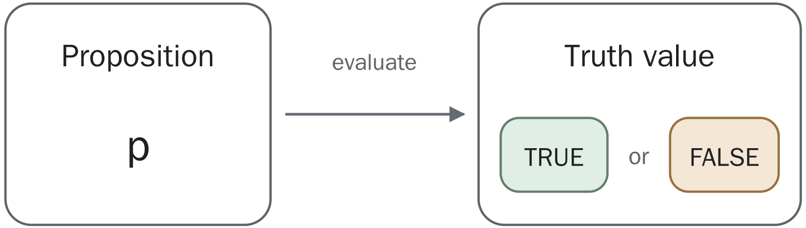
These principles have practical uses in:

- Digital circuits in computers (hardware)
- Conditional logic in programming (software)

In the following, we introduce statements and truth values, then the symbols and rules for combining them.

Propositions

- Definition & Examples



Definition: A proposition:

- is a declarative statement,
- has one definite truth value: true or false, and
- is never both true and false.

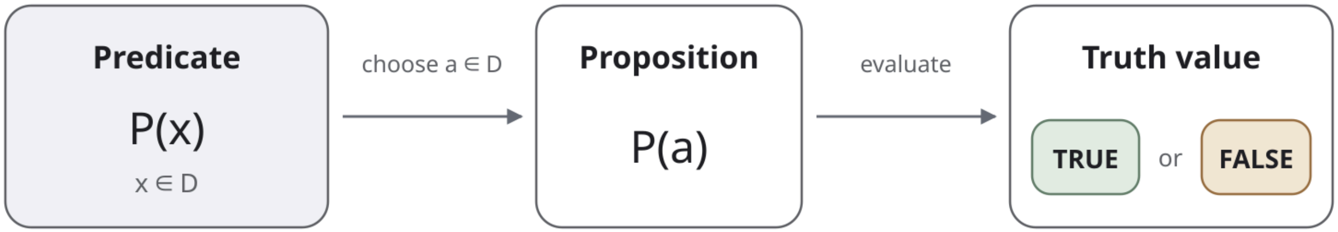
Notation: Lowercase letters such as p and q commonly denote propositions.

Examples: Evaluating the propositions below gives:

- p: “Four is even” — true
- q: “$1 + 1 = 3$” — false
- r: “$43 > 21$” — true
- s: “$4 \in \{1, 3, 5\}$” — false

Predicates

- Definition & Examples



Definition: A predicate:

- contains at least one unspecified variable,
- has a specified domain of allowed values, and
- becomes a proposition when its variables are assigned values from that domain.

Notation: Capital letters such as P and Q usually denote predicates; $P(x)$ depends on x.

Example: Suppose we are given the predicate

- $P(x)$: "x is even", where $x \in \mathbb{Z}$.

Choosing $a \in \mathbb{Z}$ produces the proposition $P(a)$:

- $P(4)$: "4 is even" — true
- $P(5)$: "5 is even" — false

Exercise Round 1

- Propositions and Predicates

Classify each item as a proposition or a predicate. State the truth value of every proposition.

1. “The capital of Denmark is Copenhagen.”

2. Classify the condition

$$x^2 < 10, \qquad x \in \mathbb{R}.$$

3. Classify the expression

$$7 \in \{1, 3, 5, 7\}.$$

4. “Nine is a prime number.”

For Exercise 5, also evaluate the specified instances.

5. Consider

$$P(n) : \quad n \text{ is divisible by } 3, \qquad n \in \mathbb{Z}.$$

- **(a)** Classify $P(n)$ as a proposition or a predicate.
- **(b)** State whether $P(12)$ and $P(14)$ are true or false.

Logical Connectives

- Overview

Connectives combine simpler propositions into compound propositions.

A compound proposition's truth value depends on those of its parts, so we consider every possible combination.

We record these cases compactly using Boolean values:

- false is 0
- true is 1

A truth table lists every possible combination of input values and the resulting output.

Symbol	Reading	When it is true
$p \wedge q$	p and q	Both are true
$p \vee q$	p or q	At least one is true
$\neg p$	Not p	p is false
$p \to q$	If p, then q	Except when p is true and q is false
$p \leftrightarrow q$	p if and only if q	Both have the same truth value

The first three also appear as the basic AND, OR, and NOT gates in digital circuits.

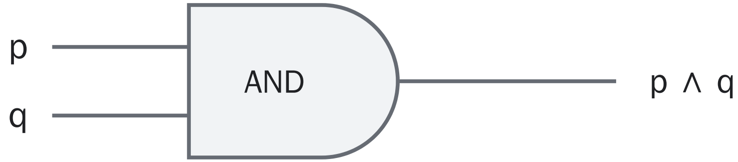
Conjunction and Disjunction

- Definitions

The conjunction $p \wedge q$ is true only when both propositions are true.

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

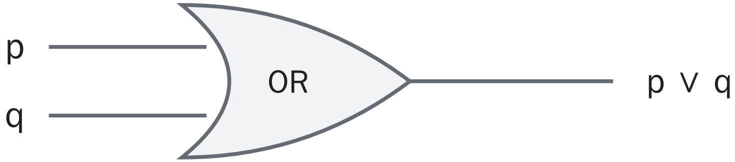
Circuit representation:



The disjunction $p \vee q$ uses inclusive "or": it is true when at least one of the two propositions is true.

p	q	$p \vee q$
0	0	0
0	1	1
1	0	1
1	1	1

Circuit representation:

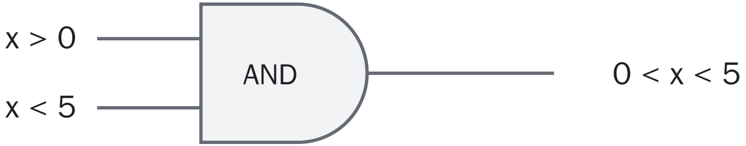


Conjunction and Disjunction

- Example: Combining Conditions

For $x \in \mathbb{R}$, combine the conditions $x > 0$ and $x < 5$ using AND. Write the result as a single equivalent condition.

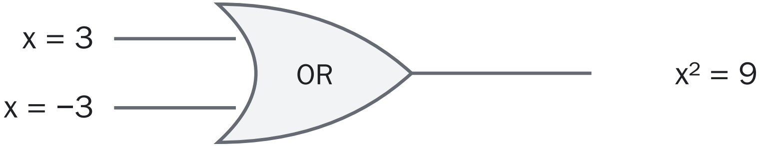
x	$x > 0$	$x < 5$	$(x > 0) \wedge (x < 5)$
3	1	1	1
0	0	1	0
5	1	0	0



Thus, $(x > 0) \wedge (x < 5)$ is equivalent to $0 < x < 5$.

For $x \in \mathbb{R}$, combine the conditions $x = 3$ and $x = -3$ using OR. Write the result as a single equivalent condition.

x	$x = 3$	$x = -3$	$(x = 3) \vee (x = -3)$
3	1	0	1
-3	0	1	1
0	0	0	0

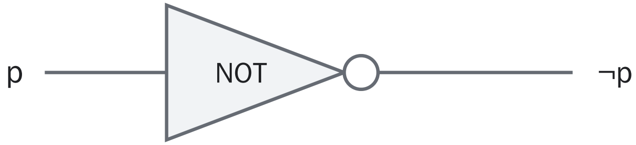


Thus, $(x = 3) \vee (x = -3)$ is equivalent to $x^2 = 9$.

Negation

- Definition & Example

p	$\neg p$
0	1
1	0



Definition: The negation of a proposition p:

- is written $\neg p$ and read "not p,"
- reverses its truth value:
  - true exactly when p is false
  - false exactly when p is true

Negating a predicate creates another predicate with the opposite truth value for each value in the domain.

Example: For $x \in \mathbb{R}$, let

- $P(x)$: "$x = 5$"
- $\neg P(x)$: "$x \neq 5$"

After choosing a value, both become propositions:

- $P(5)$: "$5 = 5$" — true, so $\neg P(5)$ is false.
- $P(7)$: "$7 = 5$" — false, so $\neg P(7)$ is true.

Thus, $\neg P(x)$ is true for every real number except 5.

Exercise Round 2

- Logical Connectives

Consider the following propositions about a customer:

- p: “A customer made a purchase”
- q: “The customer joined the mailing list”

1. Translate the following compound propositions into words:

$$p \wedge q, \qquad p \vee q, \qquad \neg p.$$

2. Suppose the customer made a purchase but did not join the mailing list. Determine the truth values in Exercise 1 for

$$p = 1, \qquad q = 0.$$

3. Construct a truth table for each compound proposition:

$$(p \wedge q) \vee \neg p, \qquad \neg p \wedge q.$$

Conditional Statements

- Definition & Example

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Definition: The conditional formed from propositions p and q:

- is written $p \to q$ and read "if p, then q,"
- has the following truth values:
  - false only when p is true and q is false
  - true in all other cases

Example: Consider the propositions

- p: "$5 > 0$"
- q: "5 is even"

The compound proposition $p \to q$ reads:

- "If $5 > 0$, then 5 is even."

Here, p is true and q is false. Thus, $p \to q$ is false.

Conditional Statements

- Example: Original, Converse, and Contrapositive

For $n \in \mathbb{Z}$, consider the predicates:

- $P(n)$: "n is divisible by 4"
- $Q(n)$: "n is even"

We begin with the original conditional.

Original: $P(n) \to Q(n)$

If n is divisible by 4, then $n = 4k$ for some $k \in \mathbb{Z}$. Since

$$n = 4k = 2(2k),$$

n is even. Thus, $P(n) \to Q(n)$ holds for every integer n. Since $P(12)$ is true, we may conclude $Q(12)$; this inference is called modus ponens.

The converse reverses the direction; the contrapositive reverses and negates both predicates.

Converse: $Q(n) \to P(n)$

If n is even, then n is divisible by 4. This fails for $n = 6$: it is even but not divisible by 4. Thus, the converse is false.

Contrapositive: $\neg Q(n) \to \neg P(n)$

Suppose n is not even. If n were divisible by 4, then $n = 4k = 2(2k)$ would be even, which is a contradiction. Hence n is not divisible by 4.

Thus, the contrapositive is also true. In general, a conditional and its contrapositive are equivalent.

Biconditionals

- Definition & Example

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$
0	0	1	1	1
0	1	0	1	0
1	0	0	0	1
1	1	1	1	1

Definition: The biconditional formed from propositions p and q:

- is written $p \leftrightarrow q$ and read "p if and only if q,"
- has the following truth values:
  - true when p and q have the same truth value
  - false when they have different truth values
- requires both $p \to q$ and $q \to p$ to be true.

Example: For $x \in \mathbb{R}$, consider the predicates

- $P(x)$: "$x = 0$"
- $Q(x)$: "$x \geq 0$ and $x \leq 0$"

Compare their truth values:

- both are true when $x = 0$
- both are false when $x \neq 0$

Thus, $P(x) \leftrightarrow Q(x)$ is true for every $x \in \mathbb{R}$.

Implication and Bi-implication

- From Particular Cases to a Whole Domain

Suppose two predicates $P(x)$ and $Q(x)$ describe conditions on the same domain. In this context, we may ask:

- whether one condition guarantees the other
- whether both conditions hold for the same values

These questions can first be examined for one chosen value.

For a particular case

Choose a from the domain. Then $P(a)$ and $Q(a)$ become propositions, and the small arrows describe this case:

$$P(a) \to Q(a), \qquad P(a) \leftrightarrow Q(a).$$

A single case, however, does not establish a relationship over the whole domain.

Across the whole domain

To describe the predicates as a whole, the relationship must hold for every allowed value of x. The large arrows express this:

$$P(x) \Rightarrow Q(x), \qquad P(x) \Leftrightarrow Q(x).$$

A bi-implication requires both directions:

$$P(x) \Leftrightarrow Q(x) \quad \text{means} \quad \begin{cases} P(x) \Rightarrow Q(x), \\ Q(x) \Rightarrow P(x). \end{cases}$$

Compare the values satisfying each condition to determine whether the relationship is one-way or two-way.

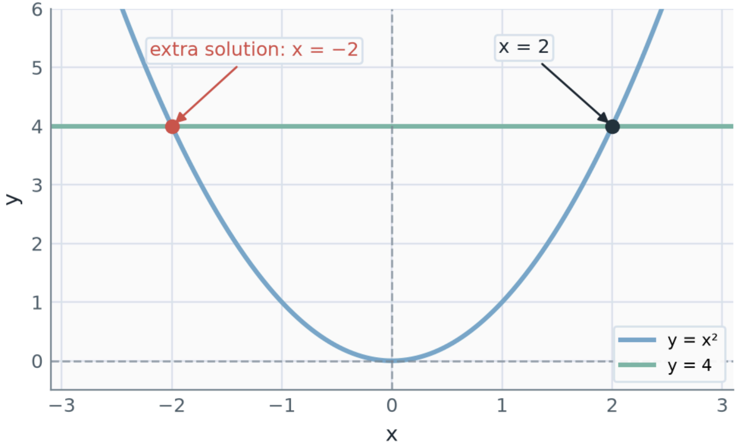
Implication and Bi-implication

- Example: One-Way and Two-Way Relationships

The graphs compare the two solution sets, revealing whether each relationship is one-way or two-way.

One direction: implication

$$x = 2 \;\Rightarrow\; x^2 = 4.$$

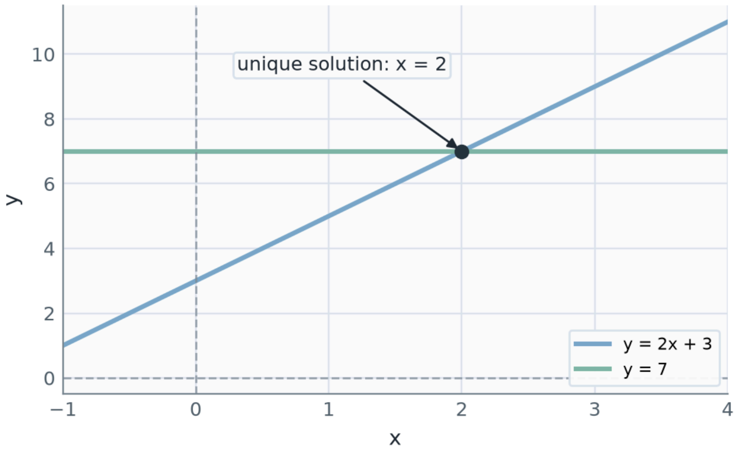


Solution sets: $\{2\} \subsetneq \{-2, 2\}$.

The extra solution $x = -2$ prevents the reverse implication, so the relationship is one-way.

Both directions: bi-implication

$$2x + 3 = 7 \;\Leftrightarrow\; x = 2.$$



Solution sets: $\{2\}$.

Therefore, both directions hold and the relationship is two-way.

Exercise Round 3

- Conditionals and Implications

1. For $x \in \mathbb{R}$, place $\Rightarrow$ or $\Leftrightarrow$ between

$$x > 4 \qquad \text{and} \qquad x^2 > 16.$$

2. For $x \in \mathbb{R}$, place $\Rightarrow$ or $\Leftrightarrow$ between

$$x^2 - 5x = 0 \qquad \text{and} \qquad x \in \{0, 5\}.$$

3. Write the converse and contrapositive of the statement:

"If an integer is a multiple of 10, then it is a multiple of 5."

Is the converse true? Justify the answer.

4. Let the propositions be:

- p: "The sensor is active"
- q: "The indicator is on"

Given the premises p and $p \to q$, state the conclusion and explain why the inference is valid.

Tautologies and Contradictions

- Definitions & Examples

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
0	1	1	0
1	0	1	0

Definitions: A logical expression is:

- a tautology if it is true in every possible case, and
- a contradiction if it is false in every possible case.

Recognizing tautologies and contradictions helps us identify statements that must be true and conditions that cannot all hold.

Examples: Consider the proposition:

- p: "The record passes validation"
- $p \vee \neg p$: "The record passes validation or it does not pass validation" — a tautology.
- $p \wedge \neg p$: "The record passes validation and it does not pass validation" — a contradiction.

Thus, the truth value of either expression is fixed, regardless of the truth value of p.

Logical Equivalence

- Definition & Examples

p	q	A: $p \vee q$	B: $q \vee p$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

p	q	A: $p \to q$	B: $\neg p \vee q$
0	0	1	1
0	1	1	1
1	0	0	0
1	1	1	1

p	q	A: $\neg(p \wedge q)$	B: $\neg p \vee \neg q$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

Definition: Let A and B denote complete logical expressions. They are logically equivalent when:

- they have the same truth value in every possible case, and
- their equivalence is written $A \equiv B$.

Logical equivalence therefore lets us rewrite or simplify statements without changing their truth value.

Observation: In all three truth tables, the columns labelled A and B match in every row.

Conclusion: Each pair of expressions is logically equivalent:

$$p \vee q \equiv q \vee p,$$
$$p \to q \equiv \neg p \vee q,$$
$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

Additional Logical Notation

- Quantifiers

Quantifiers are another part of mathematical logic. They indicate how broadly a predicate holds over its domain.

The table below provides a brief overview.

Symbol	Reading	Meaning
$\forall$	For every	The predicate holds for every allowed value
$\exists$	There exists	The predicate holds for at least one allowed value
$\exists!$	There exists exactly one	The predicate holds for one and only one value

Quantifiers are not studied further here; the aim is simply to introduce their notation, reading, and basic meaning.

Logic and Set Theory

- Matching Operations

For a fixed x, let the propositions be:

- p: “$x \in A$”
- q: “$x \in B$”

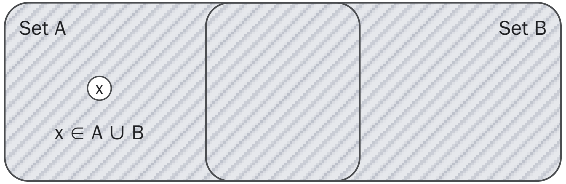
Then logical connectives correspond directly to set operations:

AND / intersection

$$p \wedge q \quad \longleftrightarrow \quad x \in A \cap B$$

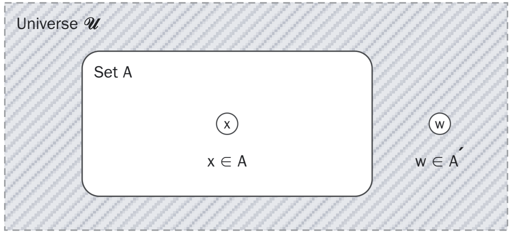
OR / union

$$p \vee q \quad \longleftrightarrow \quad x \in A \cup B$$



NOT / complement

$$\neg p \quad \longleftrightarrow \quad x \in A'$$



In each column, the logical and set-based descriptions are true for exactly the same values of x.

Exercise Round 4

- Equivalence and Logical Structure

Use a truth table or a clear logical argument where appropriate.

1. Determine whether each expression is a tautology, contradiction, or neither:

$$p \to p, \qquad p \leftrightarrow \neg p, \qquad p \vee q.$$

2. Decide whether each pair is logically equivalent:

$$\neg(p \vee q) \quad \text{and} \quad \neg p \wedge \neg q,$$
$$p \to q \quad \text{and} \quad \neg p \vee q.$$

3. For a fixed x, let the propositions be:

- p: "$x \in A$"
- q: "$x \in B$"

Express the following using set-membership notation:

$$p \wedge \neg q.$$