Mathematics Brush-Up for Data Science

Lecture Slides 12: Sequences, Sums, and Products

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Sequences, Sums, and Products

- Building Intuition

Mathematics often uses repeated patterns:

- A sequence records an ordered list, such as $1, 2, 3, 4, \ldots$
- A sum repeatedly adds terms, such as $1 + 2 + 3 + 4 + \ldots$
- A product repeatedly multiplies terms, such as $1 \cdot 2 \cdot 3 \cdot 4 \cdots$

Writing every term becomes impractical when an expression contains many terms or has a variable length.

Compact indexed notation lets us specify:

- The pattern followed by the terms
- Where the index starts and stops
- Whether the terms are listed, added, or multiplied

We first introduce notation for sequences, then apply the same indexed structure to sums and products.

Sequences

- Definition

A sequence is an ordered list of numbers:

$$0, 1, 2, 3, 4, 5, \ldots$$

Changing the order generally produces a different sequence.

An indexed sequence is written as:

$$a_0, a_1, a_2, a_3, \ldots$$

Here:

- a_n denotes the term with index n
- The index may begin at 0 or 1

To denote the sequence as a whole, we write

$$(a_n)_{n=0}^{\infty}$$

when the index range should be explicit, or

$$(a_n)$$

when the range is clear.

Braces are also used:

$$\{a_n\}_{n=0}^{\infty} \quad \text{or} \quad \{a_n\}.$$

However, braces may be mistaken for set notation; parentheses emphasize that order matters.

Sequences

- Finite & Infinite Sequences

A finite sequence

A sequence with exactly n terms is called a finite sequence, or an n-tuple:

$$(a_i)_{i=1}^{n} = (a_1, a_2, \ldots, a_n).$$

Its length is n.

For example, let $n = 1, 2, 3, 4$ and $a_n = 2n$, then

$$(a_n)_{n=1}^{4} = (2, 4, 6, 8),$$

which is a 4-tuple of length 4.

An infinite sequence

A sequence with infinitely many terms is called an infinite sequence:

$$(a_n)_{n=1}^{\infty} = (a_1, a_2, a_3, \ldots).$$

It has no finite length and is not an n-tuple.

For example, let $n = 1, 2, 3, \ldots$ and $a_n = 2n$, then

$$(a_n)_{n=1}^{\infty} = (2, 4, 6, 8, \ldots).$$

Sequences

- Examples: Recorded Daily Values

Customer orders

A small store records its number of orders on four consecutive days:

$$(a_d)_{d=1}^4 = (18, 25, 20, 27).$$

- d is the day index
- a_d is the number of orders on day d

For example, $a_3 = 20$ means that the store received 20 orders on day 3.

Daily rainfall

A weather station records rainfall, in millimetres, over the same four days:

$$(r_d)_{d=1}^4 = (2, 0, 5, 1).$$

- d is the day index
- r_d is the rainfall, in millimetres, on day d

For example, $r_3 = 5\ \mathrm{mm}$ means that $5\ \mathrm{mm}$ of rain fell on day 3.

Sequences & Tuples

- Tuples Are Not Sets

Consider the tuple:

$$(2, 5, 5)$$

The tuple has three entries, each with a fixed position.

- Repeated entries are retained:

$$(2, 5, 5) \neq (2, 5)$$

- Order matters:

$$(2, 5, 5) \neq (5, 2, 5)$$

Consider the set:

$$\{2, 5, 5\}$$

The set contains two distinct elements 2 and 5.

- Repeated listings are ignored:

$$\{2, 5, 5\} = \{2, 5\}$$

- Order does not matter:

$$\{2, 5\} = \{5, 2\}$$

Finite Sequences & Coordinates

- Coordinate Spaces

When every entry of an n-tuple is a real number,

$$(x_i)_{i=1}^{n} = (x_1, x_2, \ldots, x_n).$$

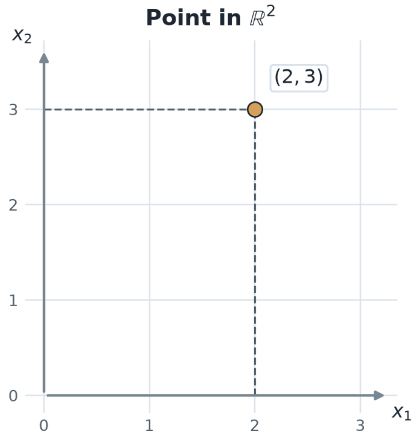
the tuple belongs to the coordinate space

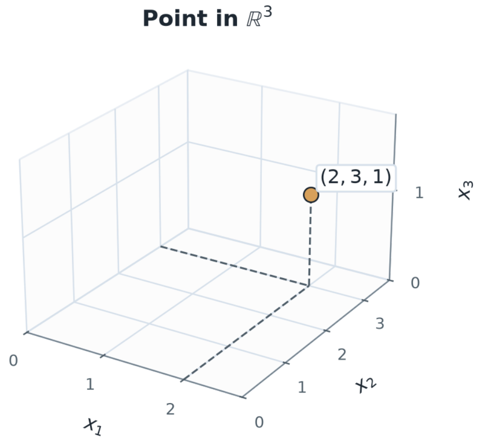
$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies}}.$$

The tuple $(2, 3) \in \mathbb{R}^2$ represents a point in the plane.

The tuple $(2, 3, 1) \in \mathbb{R}^3$ represents a point in three-dimensional space.

Here, each tuple identifies a point. In linear algebra, the same tuple can give the coordinates of a vector; context determines the interpretation.





Exercise Round 1

- Sequences & Tuples

1. Write the first five terms of $(a_n)_{n=1}^{\infty}$ when

$$a_n = 2n - 1.$$

2. Decide which statements are true:

$$\begin{aligned}(1,2,1) &= (1,1,2),\\ (1,2,1) &= (1,2),\\ \{1,2,1\} &= \{1,1,2\},\\ (1,2,1) &= \{1,2\}.\end{aligned}$$

3. Let $p = (-2, 3, 1) \in \mathbb{R}^3$. Find p_1, p_2, and p_3. Then replace the second coordinate by 5 and write the resulting tuple.

Summation Notation

- Definition

Given a sequence (a_n) and integers $m \leq p$,

$$\sum_{n=m}^{p} a_n = a_m + a_{m+1} + \cdots + a_p.$$

- $\sum$ is the summation symbol, based on capital Greek sigma
- The symbol is read "the sum"
- a_n is the summand, or term being added
- n is the index; within the sum, it is a dummy variable and may be renamed consistently
- m and p are the lower and upper limits

The index tells us which terms to generate, while the limits tell us where to start and stop.

Summation Notation

- Examples: Totals from Daily Records

Total customer orders

For the customer-order sequence

$$(a_d)_{d=1}^{4} = (18, 25, 20, 27).$$

The total number of orders is

$$\begin{aligned}\sum_{d=1}^{4} a_d &= a_1 + a_2 + a_3 + a_4 \\ &= 18 + 25 + 20 + 27 \\ &= 90.\end{aligned}$$

Total rainfall

The rainfall sequence, measured in millimetres, is

$$(r_d)_{d=1}^{4} = (2, 0, 5, 1).$$

The total rainfall is

$$\begin{aligned}\sum_{d=1}^{4} r_d &= r_1 + r_2 + r_3 + r_4 \\ &= 2 + 0 + 5 + 1 \\ &= 8\ \text{mm}.\end{aligned}$$

Summation Notation

- Example: Expanding and Writing Sums

Expand the sum by substituting $n = 1, 2, 3, \ldots, 100$:

$$\sum_{n=1}^{100}(3+4n) = (3+4\cdot 1) + (3+4\cdot 2) + \cdots$$
$$+ (3+4\cdot 100)$$
$$= 7 + 11 + 15 + \cdots + 403.$$

Write the sum, by identifying each term and its index:

$$\underbrace{\frac{1}{1}}_{n=1} + \underbrace{\frac{1}{2}}_{n=2} + \underbrace{\frac{1}{3}}_{n=3} + \cdots + \underbrace{\frac{1}{50}}_{n=50} = \sum_{n=1}^{50}\frac{1}{n}.$$

Expand the sum for a variable upper limit:

$$\sum_{n=1}^{p}\left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^p$$
$$= \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^p}.$$

Similarly, for the alternating sum:

$$\underbrace{1}_{n=1} - \underbrace{2}_{n=2} + \underbrace{3}_{n=3} - \cdots - \underbrace{8}_{n=8} = \sum_{n=1}^{8}(-1)^{n-1}n.$$

Summation Notation

- Example: Evaluating Sums

Expand and evaluate:

$$\sum_{g=4}^{9}(5g+3) = (5\cdot 4+3)+(5\cdot 5+3)+\cdots$$
$$+(5\cdot 9+3)$$
$$= 23+28+33+38+43+48$$
$$= 213.$$

Similarly, for powers of 10:

$$\sum_{k=1}^{3}\frac{7}{10^k} = \frac{7}{10^1}+\frac{7}{10^2}+\frac{7}{10^3}$$
$$= \frac{7}{10}+\frac{7}{100}+\frac{7}{1000}$$
$$= 0.777.$$

When the number of terms is manageable, direct expansion is often the clearest method.

Summation Notation

- Properties

Let c be a constant and let (a_k) and (b_k) be sequences.

Combining terms

Term-by-term addition:

$$\sum_{k=m}^{p}(a_k + b_k) = \sum_{k=m}^{p} a_k + \sum_{k=m}^{p} b_k.$$

Constant multiple:

$$\sum_{k=m}^{p} ca_k = c\sum_{k=m}^{p} a_k.$$

Ranges and constants

Sum of a constant:

$$\sum_{k=m}^{p} c = (p - m + 1)c.$$

Splitting a sum, where $m \leq q < p$:

$$\sum_{k=m}^{p} a_k = \sum_{k=m}^{q} a_k + \sum_{k=q+1}^{p} a_k.$$

When it comes to index shifts: Both the limits and the summand must change consistently.

Summation Notation

- Example: Applying and Checking Properties

Use the properties to evaluate the following sum:

$$\begin{aligned}\sum_{k=1}^{4}(2k+3) &= \sum_{k=1}^{4}2k + \sum_{k=1}^{4}3 \\ &= 2\sum_{k=1}^{4}k + 4\cdot 3 \\ &= 2(1+2+3+4)+12 \\ &= 32.\end{aligned}$$

Check the result by expanding the original sum directly:

$$\begin{aligned}\sum_{k=1}^{4}(2k+3) &= (2\cdot 1+3)+(2\cdot 2+3) \\ &\quad +(2\cdot 3+3)+(2\cdot 4+3) \\ &= 5+7+9+11 \\ &= 32.\end{aligned}$$

Both routes give the same value. Direct expansion illustrates why the properties work in this example.

Exercise Round 2

- Summation Notation

1. Expand the following sum:

$$\sum_{k=0}^{n} 2^k.$$

2. Rewrite the following using summation notation:

$$2 + 4 + 6 + 8 + \cdots + 2n.$$

3. Use the summation properties to evaluate:

$$\sum_{k=1}^{5} (3k - 2).$$

4. Use the summation splitting property to split the following sum after its fourth term:

$$\sum_{k=1}^{10} a_k.$$

Product Notation

- Definition

Given a sequence (a_n) and integers $m \leq p$,

$$\prod_{n=m}^{p} a_n = a_m \cdot a_{m+1} \cdots a_p.$$

- $\prod$ is the product symbol, based on capital Greek pi
- The symbol is read “the product”
- a_n is the factor, or term being multiplied
- n is the index; within the product, it is a dummy variable and may be renamed consistently
- m and p are the lower and upper limits

The index tells us which factors to generate, while the limits tell us where to start and stop.

Product Notation

- Examples: Counting Combinations

Possible outfits

Suppose an outfit contains one item from each category:

$$(a_i)_{i=1}^{3} = (3, 2, 4),$$

representing 3 shirts, 2 pairs of trousers, and 4 pairs of shoes. The number of possible outfits is

$$\prod_{i=1}^{3} a_i = 3 \cdot 2 \cdot 4 = 24.$$

Possible identification codes

A code contains two letters followed by three digits, and symbols may repeat. The numbers of choices at its five positions are

$$(c_i)_{i=1}^{5} = (26, 26, 10, 10, 10).$$

The number of possible codes is

$$\prod_{i=1}^{5} c_i = 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 = 676{,}000.$$

Product Notation

- Example: Expanding and Writing Products

Expand by substituting every index value:

$$\prod_{n=1}^{4} n = 1 \cdot 2 \cdot 3 \cdot 4 = 24.$$

Expand the product for a variable upper limit:

$$\begin{aligned}\prod_{n=0}^{p}(2+3n) &= (2+3\cdot 0)(2+3\cdot 1)\cdots(2+3p)\\ &= 2\cdot 5\cdot 8\cdots(2+3p).\end{aligned}$$

Identify each factor and its index:

$$\underbrace{2}_{n=1} \cdot \underbrace{4}_{n=2} \cdot \underbrace{6}_{n=3} \cdots \underbrace{2p}_{n=p} = \prod_{n=1}^{p} 2n.$$

Similarly, for the odd factors:

$$\underbrace{1}_{n=1} \cdot \underbrace{3}_{n=2} \cdot \underbrace{5}_{n=3} \cdots \underbrace{(2p-1)}_{n=p} = \prod_{n=1}^{p} (2n-1).$$

Product Notation

- Example: Evaluating Products

A changing factor

Substitute each index value before evaluating the factors:

$$\begin{aligned}\prod_{n=1}^{3}(n+1) &= (1+1)(2+1)(3+1)\\ &= 2\cdot 3\cdot 4\\ &= 24.\end{aligned}$$

A constant factor

Substitute each index value; the same factor appears each time:

$$\begin{aligned}\prod_{k=1}^{4}5 &= 5\cdot 5\cdot 5\cdot 5\\ &= 5^4\\ &= 625.\end{aligned}$$

Product Notation

- Properties

Let c be a constant and let (a_k) and (b_k) be sequences.

Combining factors

Term-by-term multiplication:

$$\prod_{k=m}^{p}(a_k b_k) = \left(\prod_{k=m}^{p} a_k\right)\left(\prod_{k=m}^{p} b_k\right).$$

Constant factor:

$$\prod_{k=m}^{p} c a_k = c^{p-m+1}\prod_{k=m}^{p} a_k.$$

Ranges and constants

Product of a constant:

$$\prod_{k=m}^{p} c = c^{p-m+1}.$$

Splitting a product, where $m \le q < p$:

$$\prod_{k=m}^{p} a_k = \left(\prod_{k=m}^{q} a_k\right)\left(\prod_{k=q+1}^{p} a_k\right).$$

When it comes to index shifts: Both the limits and the factor must change consistently.

Factorials & Empty Index Ranges

- Identity Values

Factorial notation

For a positive integer n,

$$n! = 1 \cdot 2 \cdots n = \prod_{k=1}^{n} k.$$

Empty sum

When $m > p$, the index range is empty, so

$$\sum_{n=m}^{p} a_n = 0.$$

Here, 0 is the additive identity.

Empty product

When $m > p$, the index range is empty, so

$$\prod_{n=m}^{p} a_n = 1.$$

Here, 1 is the multiplicative identity.

Applying the empty-product identity to the factorial formula gives:

$$0! = \prod_{k=1}^{0} k = 1.$$

Indexed Unions & Intersections

- Repeated Set Operations

There are set-based analogues of compact sum and product notation:

The indexed union of sets $A_1, A_2, \ldots, A_n$:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n.$$

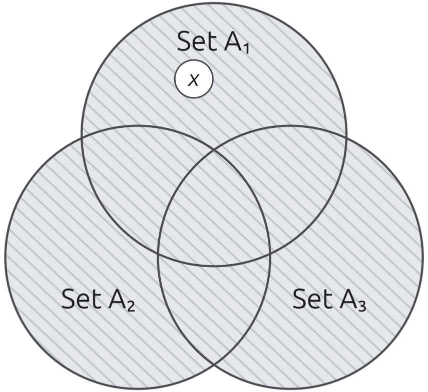
contains every element that belongs to at least one A_i

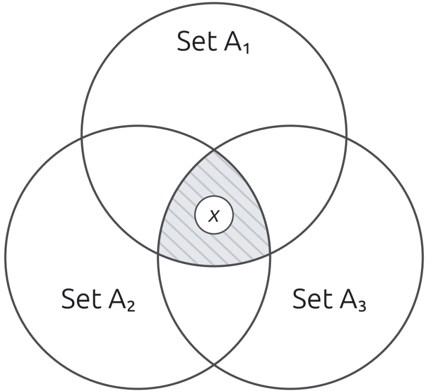
.

The indexed intersection of sets $A_1, A_2, \ldots, A_n$:

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n.$$

contains every element that belongs to every A_i.





Indexed Unions & Intersections

- Example: Combining Several Sets

Consider the indexed sets:

$$B_1 = \{1, 2, 3\}, \qquad B_2 = \{2, 3, 4\}, \qquad B_3 = \{3, 4, 5\}.$$

Indexed union

Collect the elements that occur in at least one set:

$$\begin{aligned} \bigcup_{i=1}^{3} B_i &= B_1 \cup B_2 \cup B_3 \\ &= \{1, 2, 3, 4, 5\}. \end{aligned}$$

Indexed intersection

Keep the elements that occur in every set:

$$\begin{aligned} \bigcap_{i=1}^{3} B_i &= B_1 \cap B_2 \cap B_3 \\ &= \{3\}. \end{aligned}$$

Sums & Products over Sets

- Example: Aggregating a Finite Collection

Let $S = \{2, 4, 7\}$ and define $f(x) = x + 1$.

Sum over the set

Evaluate $f(x)$ for each $x \in S$, then add:

$$\begin{aligned}\sum_{x \in S} f(x) &= f(2) + f(4) + f(7) \\ &= 3 + 5 + 8 \\ &= 16.\end{aligned}$$

Product over the set

Evaluate $f(x)$ for each $x \in S$, then multiply:

$$\begin{aligned}\prod_{x \in S} f(x) &= f(2) \cdot f(4) \cdot f(7) \\ &= 3 \cdot 5 \cdot 8 \\ &= 120.\end{aligned}$$

Exercise Round 3

- Products & Indexed Operations

1. Expand and simplify:

$$\prod_{k=2}^{5} \frac{k}{k-1}.$$

2. Use the product properties to rewrite using factorial notation:

$$\prod_{k=1}^{n} 3k.$$

3. Split the product after its third factor:

$$\prod_{k=1}^{8} a_k.$$

4. Find the indexed union and intersection for

$$A_1 = \{1,2\}, \quad A_2 = \{2,3\}, \quad A_3 = \{2,4\}.$$

$$\bigcup_{i=1}^{3} A_i \qquad \text{and} \qquad \bigcap_{i=1}^{3} A_i.$$

5. For $S = \{1,2,3\}$, evaluate:

$$\sum_{x \in S} x^2.$$