Mathematics Brush-Up for Data Science

Lecture Slides 11: Multivariable Functions

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Multivariable Functions

- Building Intuition

We have focused on functions of a single variable:

- Each input is a single number x
- Each output is a single number $f(x)$

Many situations, however, involve relationships between more than one independent variable.

Examples:

- Temperature:
  - depends on latitude and longitude
- Company profit:
  - depends on units sold and unit price
- Crop yield:
  - depends on fertilizer, rainfall, soil pH, etc.

Domains of (real-valued) functions:

- Single-variable functions $f(x)$
  - Domain: a subset of the number line $\mathbb{R}$, often one or more intervals
- Two-variable functions $f(x_1, x_2)$
  - Domain: a subset of the plane $\mathbb{R}^2$, often a region
- Three-variable functions $f(x_1, x_2, x_3)$
  - Domain: a subset of space $\mathbb{R}^3$
- n-variable functions $f(x_1, x_2, \ldots, x_n)$
  - Domain: a subset of $\mathbb{R}^n$

As we add more variables, the domain lives in higher-dimensional spaces.

A multivariable function maps points located in these higher-dimensional spaces to the real numbers $\mathbb{R}$.

Multivariable Functions

- Definition

More formally, a real-valued function of n variables is a rule that assigns to each input:

$$(x_1, x_2, \ldots, x_n) \in A \subseteq \mathbb{R}^n$$

exactly one real number:

$$f(x_1, x_2, \ldots, x_n) \in \mathbb{R}$$

This is written as:

$$f : A \to \mathbb{R}$$

The range (or image) of f is the set of all actual outputs:

$$\mathrm{Range}(f) = \{f(x_1, x_2, \ldots, x_n) \mid (x_1, x_2, \ldots, x_n) \in A\}$$

Multivariable Functions

- Visualizing Functions of Multiple Variables

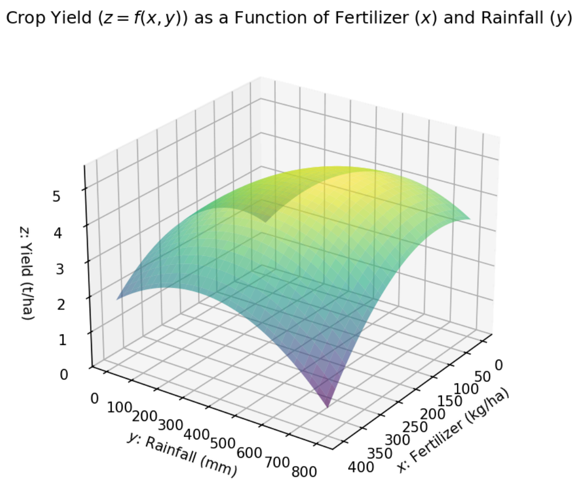
Note that, when we want to visualize:

- A single-variable function, then:
  - The graph of $f(x)$ lies in $\mathbb{R}^2$
  - A plane: input axis + output axis
- two-variable function, then:
  - The graph of $f(x_1, x_2)$ lies in $\mathbb{R}^3$
  - A 3D space: two input axes + output axis
- A function depending on three or more variables:
  - The graph of $f(x_1, x_2, \ldots, x_n)$ lies in $\mathbb{R}^{n+1}$

We can not visualize such functions directly.

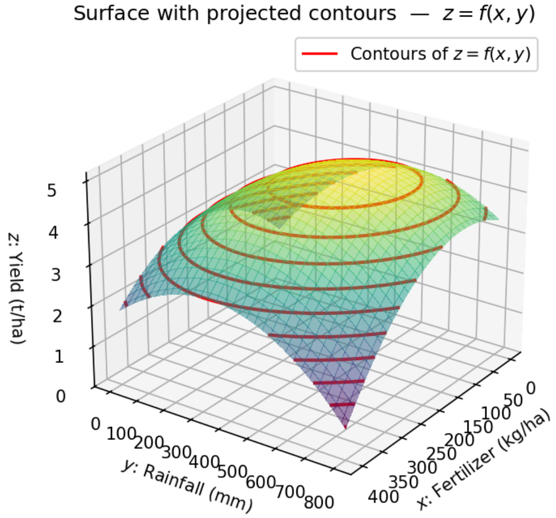
We need projections, level sets, and slices to study their behavior.

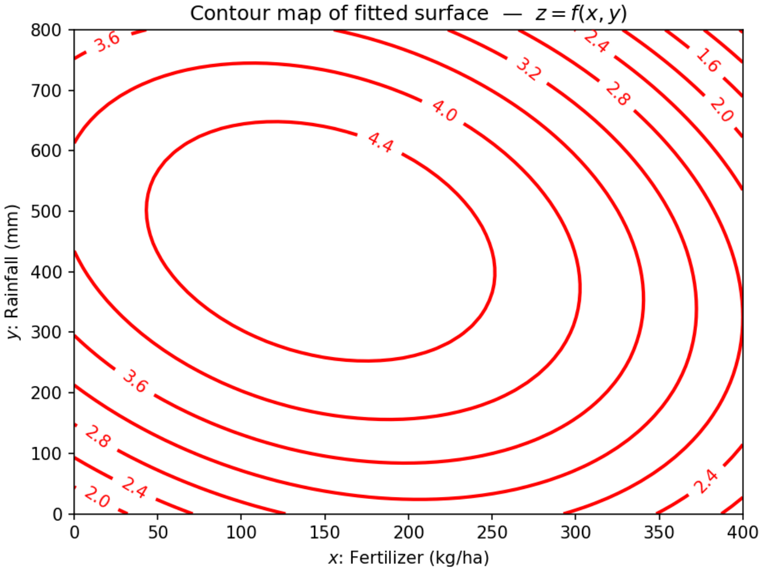
To illustrate these techniques, we extend the crop yield model we studied earlier to include additional factors:



Level Curves/Contours

- Definition





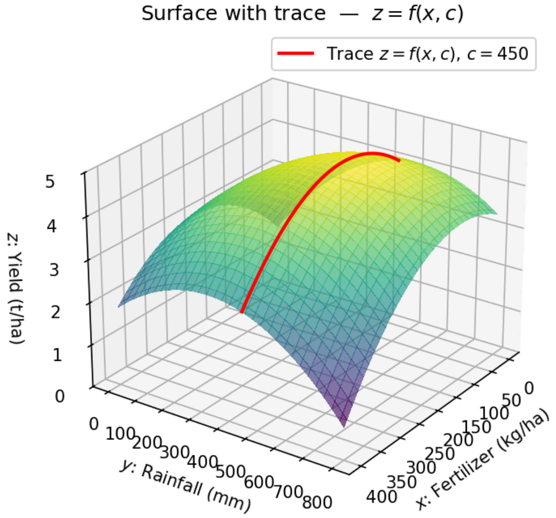
For $f(x, y)$, a level curve (or contour) is the set of inputs that produce a fixed output k:

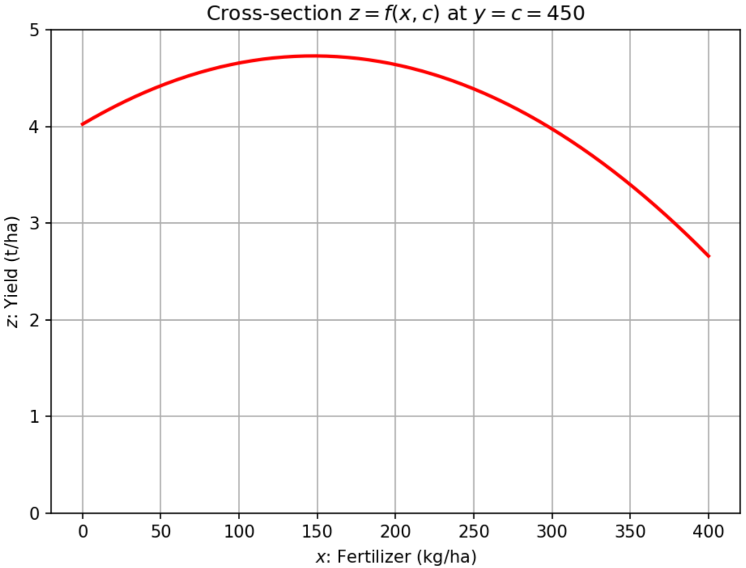
$$f(x, y) = k$$

In the xy-plane, it connects equal-output inputs. In the crop model, this reveals fertilizer–rainfall trade-offs that give the same yield.

Function Traces

- Varying x w





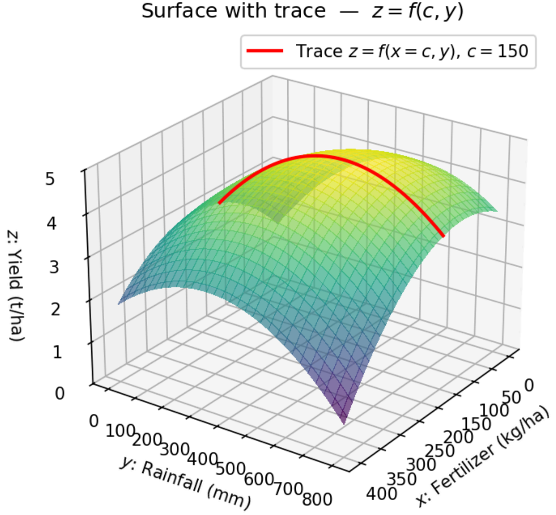
A trace fixes one variable and lets the other vary. For a trace in the x-direction, fix $y = c$:

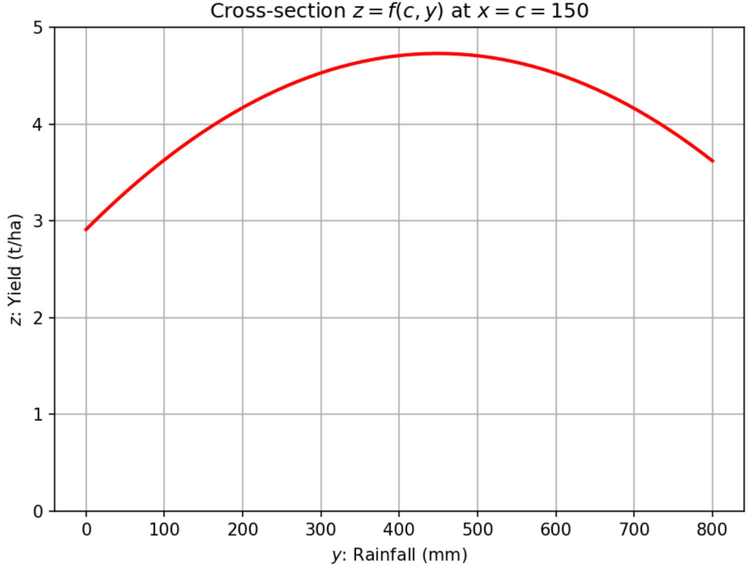
$$z = f(x, c)$$

This curve lies in a vertical plane parallel to the xz-plane. Here rainfall is fixed at 450 mm while fertilizer varies.

Function Traces

- Varying y w





A trace fixes one variable and lets the other vary. For a trace in the y-direction, fix $x = c$:

$$z = f(c, y)$$

This curve lies in a vertical plane parallel to the yz-plane. Here fertilizer is fixed at $150\ \mathrm{kg/ha}$ while rainfall varies.

Exercise Round 1

- Level Curves and Traces

Consider the function

$$f(x, y) = x^2 + y^2.$$

1. Find and sketch the level curve:

$$f(x, y) = 4$$

2. Find the trace obtained by fixing the following value:

$$y = 1$$

Then describe its graph.