Mathematics Brush-Up for Data Science

Lecture Slides 10: Differentiation

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Derivatives

What is a derivative?

- A derivative tells us the slope of the function at a single point on its graph

Central to the concept of a derivative is the concept of:

- Secant lines (slopes over an interval)
- Tangent lines (slopes at a single point)

These ideas allow us to quantify how a function changes.

Secant Lines

Suppose that we have:

- A function f
- A point a
- Another point $a+h$ nearby ($h \neq 0$)

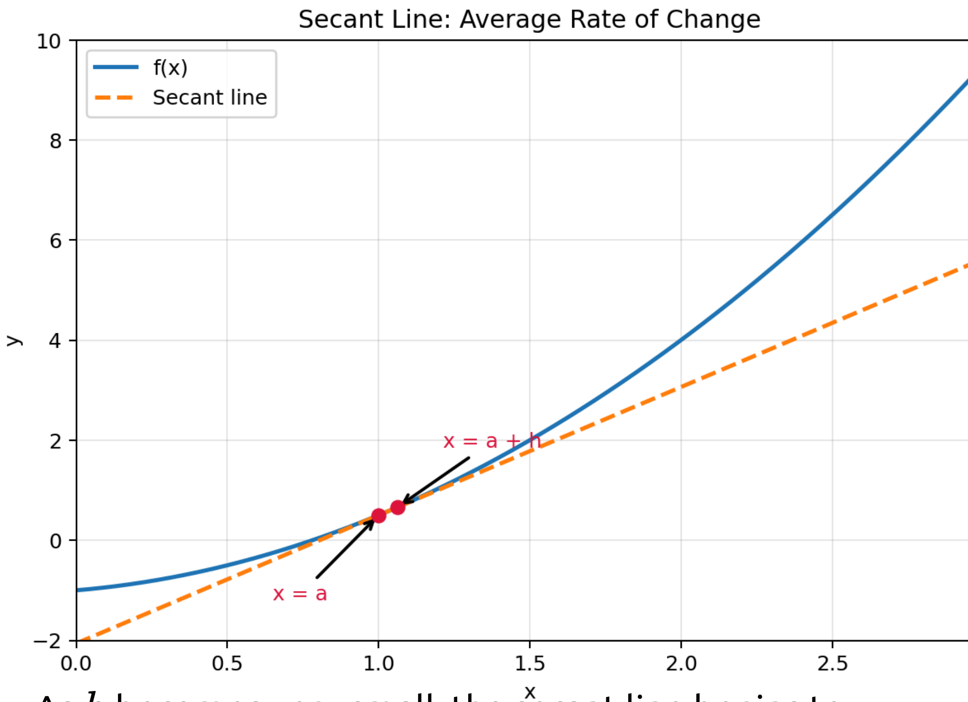
In this case, the slope of the secant line between

$$(a, f(a)) \quad \text{and} \quad (a+h, f(a+h))$$

represents the average rate of change of f on the interval $[a, a+h]$.

The slope of this line is:

$$m_{\text{sec}} = \frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}$$



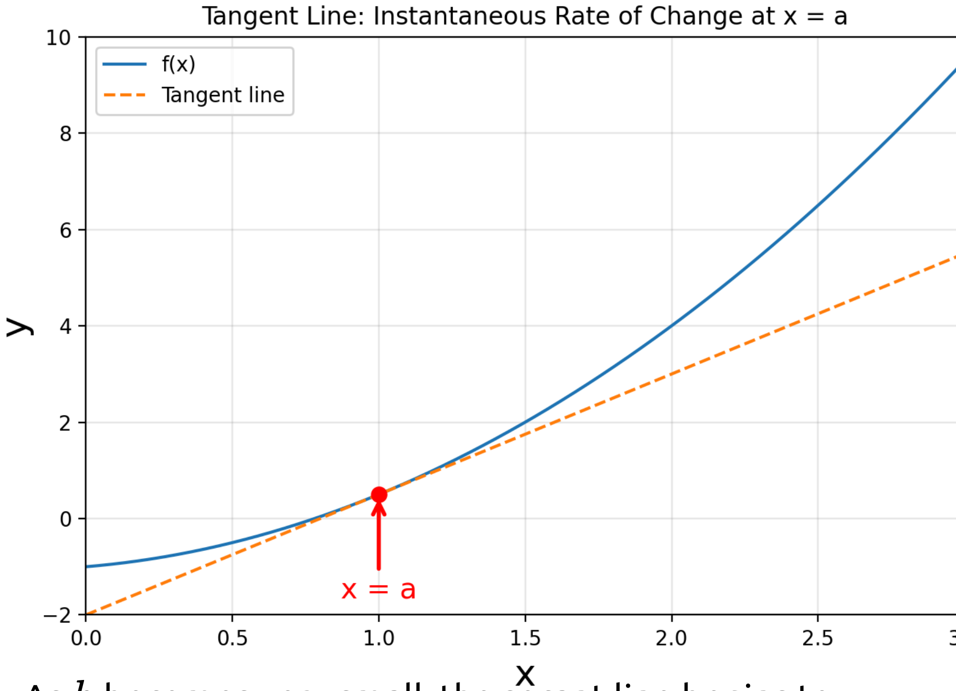
As h becomes very small, the secant line begins to "approach" the tangent line at a.

Tangent Lines

The tangent line is the "limit position" of the secant line as the second point approaches the first:

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit actually exists.



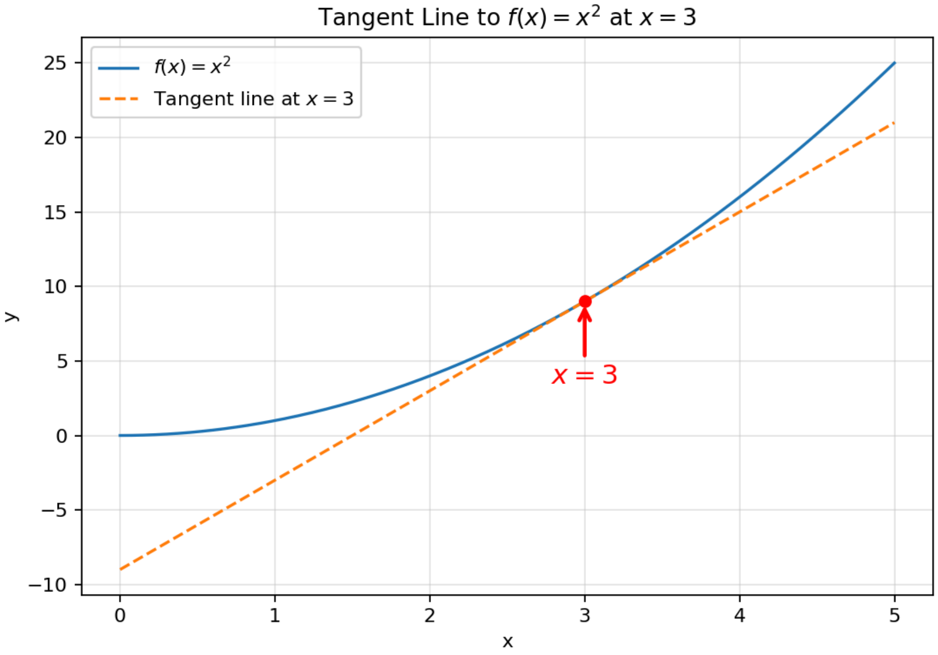
As h becomes very small, the secant line begins to "approach" the tangent line at a.

Tangent Lines

- Example: Finding a Tangent Slope

Find the slope of the tangent line to the graph of $f(x) = x^2$ at the point $x = 3$.

$$\begin{aligned} m_{tan} &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} \\ &= \lim_{h\to 0} \frac{(3+h)^2 - 9}{h} \\ &= \lim_{h\to 0} \frac{9 + 6h + h^2 - 9}{h} \\ &= \lim_{h\to 0} \frac{h(6+h)}{h} \\ &= \lim_{h\to 0} 6 + h \\ &= 6 \end{aligned}$$



The Derivative of a Function

- Definition

To define the derivative of a function, let:

- $I \subseteq \mathbb{R}$ be an open interval containing a
- $f : I \to \mathbb{R}$ be a function

The derivative of f at a is then defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

Note that:

- If $f'(a)$ exists, we say f is differentiable at a.
- If $f'(x)$ exists for every $x \in I$, we say f is differentiable on I.

Instead of writing $f'(x)$ we sometimes write:

$$\frac{df}{dx} \quad \text{or} \quad \frac{d}{dx} f(x)$$

The Derivative of a Function

- Example: Linear Functions

Consider the linear function:

$$f(x) = ax + b,\ a \neq 0,\ b \in \mathbb{R},$$

for any $x \in \mathbb{R}$.

We compute the derivative of the function, as follows:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(a(x+h) + b) - (ax + b)}{h} \\ &= \lim_{h \to 0} \frac{(ax + ah + b) - (ax + b)}{h} \\ &= \lim_{h \to 0} \frac{ax + ah + b - ax - b}{h} \\ &= \lim_{h \to 0} \frac{ah}{h} \\ &= \lim_{h \to 0} a \\ &= a \end{aligned}$$

Intuitively, this makes sense, as a linear function is a straight line with constant slope.

The Derivative of a Function

- Example: Quadratic Functions

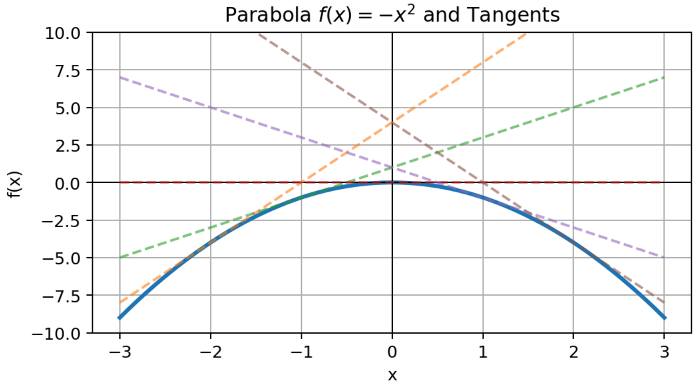
Similarly, consider the quadratic function:

$$f(x) = ax^2,\ a \neq 0,$$

for any $x \in \mathbb{R}$.

We compute the derivative of the function, as follows:

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{a(x+h)^2 - ax^2}{h} \\
&= a \cdot \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\
&= a \cdot \lim_{h \to 0} \frac{x^2 + h^2 + 2hx - x^2}{h} \\
&= a \cdot \lim_{h \to 0} \frac{h^2 + 2hx}{h} \\
&= a \cdot \lim_{h \to 0} \frac{h(h+2x)}{h} \\
&= a \cdot \lim_{h \to 0} (h + 2x) \\
&= 2ax
\end{aligned}$$



- If $a < 0$ and $x > 0$ the slope is negative
- If $a < 0$ and $x < 0$ the slope is positive

Exercise Round 1

- The Limit Definition

Use the limit definition of the derivative. For each problem, simplify the difference quotient before taking the limit.

1. Find the slope of the tangent line at the stated input.

$$f(x) = x^3, \qquad x = -1.$$

2. Find the derivative function.

$$f(x) = x^2 - 2x + 1.$$

Some Common Derivatives

Below is a table of some of the most frequently used derivatives.

Function $f(x)$	Derivative $f'(x)$	Notes
k	0	Constant rule; $k \in \mathbb{R}$
x^n	nx^{n-1}	Power rule; $n \in \mathbb{R} \setminus \{0\}$; where x^n is real-valued and differentiable
$\ln(x)$	$\frac{1}{x}$	$x > 0$
e^x	e^x	—
a^x	$\ln(a) \cdot a^x$	$a > 0$
$\sin(x)$	$\cos(x)$	Angles in radians
$\cos(x)$	$-\sin(x)$	Angles in radians

Common Differentiation Rules

Just like for limits, there are certain rules that we can apply when differentiating functions. Let f and g be differentiable on an interval. For the quotient rule, additionally require $g(x) \neq 0$ at the point under consideration.

Operation	Formula
Constant Multiple	$\frac{d}{dx}\big(kf(x)\big) = kf'(x), \quad k \in \mathbb{R}$
Sum	$\frac{d}{dx}\big(f(x) + g(x)\big) = f'(x) + g'(x)$
Difference	$\frac{d}{dx}\big(f(x) - g(x)\big) = f'(x) - g'(x)$
Product	$\frac{d}{dx}\big(f(x) \cdot g(x)\big) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{\big(g(x)\big)^2}, \quad g(x) \neq 0$

Common Differentiation Rules

- Examples: Product and Quotient Rules

Differentiate the functions:

- $f(x) = 2x^2$
- $g(x) = x$

Then consider the derivative of:

- The product $f(x) \cdot g(x)$
- The quotient $\frac{f(x)}{g(x)}$

We get from the table of derivatives that:

$$f'(x) = \frac{d}{dx}(2x^2) = 4x$$

$$g'(x) = \frac{d}{dx}(x) \quad = 1$$

Now, computing the product, we get:

$$\begin{aligned}\frac{d}{dx}(f(x) \cdot g(x)) &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\ &= 4x \cdot x + 2x^2 \cdot 1 \\ &= 6x^2\end{aligned}$$

For the quotient, assume $x \neq 0$, we get:

$$\begin{aligned}\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{\big(g(x)\big)^2} \\ &= \frac{4x \cdot x - 2x^2 \cdot 1}{x^2} \\ &= \frac{2x^2}{x^2} \\ &= 2\end{aligned}$$

Common Differentiation Rules

- Example: Applying the Product Rule

Compute the derivative of the function:

$$h(x) = e^x \cdot \sin(x)$$

We notice that this is a product of functions.

Identifying constituent parts:

- $f(x) = e^x$
- $g(x) = \sin(x)$

In the product:

$$h(x) = f(x) \cdot g(x)$$

This means that we can use the product rule to compute the derivative of $h(x)$.

The product rule gives us:

$$\begin{aligned} h'(x) &= \frac{d}{dx}\big((f(x) \cdot g(x))\big) \\ &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \end{aligned}$$

From the table of derivatives we get that:

- $f'(x) = e^x$
- $g'(x) = \cos(x)$

Substituting in our results we get:

$$\begin{aligned} h'(x) &= e^x \cdot \sin(x) + e^x \cdot \cos(x) \\ &= e^x \cdot (\sin(x) + \cos(x)) \end{aligned}$$

Exercise Round 2

- Differentiation Rules

Differentiate using the derivative table and the appropriate differentiation rules:

1. $f(x) = 2x^5 + 7$
2. $f(x) = \dfrac{6}{x^2}$
3. $f(x) = \dfrac{3x+1}{4x-3}$
4. $f(t) = t \cdot e^t$

The Chain Rule

What is the chain rule?

- It is a rule for finding the derivative of the composition of two or more functions

Let f and g be functions such that:

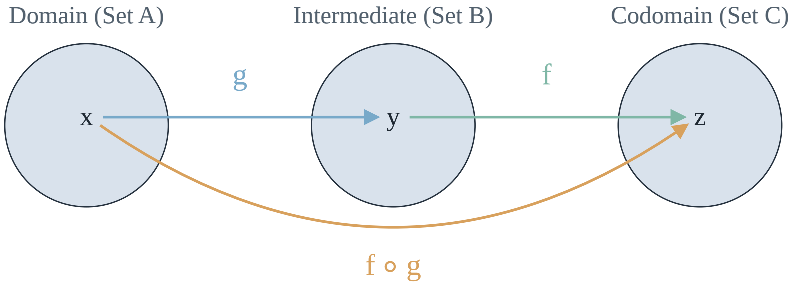
- g is differentiable at x
- f is differentiable at $g(x)$

For the composite function:

$$h(x) = (f \circ g)(x) = f(g(x))$$

the derivative is then defined as:

$$\frac{d}{dx}(f \circ g)(x) = f'(g(x)) \cdot g'(x)$$



To differentiate $h(x) = f(g(x))$, follow the steps:

1. Identify outer function f and inner function g
2. Differentiate f with respect to its argument u to get $f'(u)$
3. Substitute $u = g(x)$ into $f'(u)$ to obtain $f'(g(x))$
4. Differentiate g with respect to its argument x to get $g'(x)$
5. Multiply them to find:

$$h'(x) = f'(g(x)) \cdot g'(x)$$

The Chain Rule

- Examples: Applying the Chain Rule

Differentiate $h(x) = (\sin(x))^2$.

To do so, we let:

- The outer function be
$$f(u) = u^2, \text{ so } f'(u) = 2u$$
- The inner function be
$$g(x) = \sin(x), \text{ so } g'(x) = \cos(x)$$

Applying the Chain Rule, we then get:

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= 2\sin(x) \cdot \cos(x) \end{aligned}$$

Differentiate $h(x) = e^{x^3+1}$.

To do so, we let:

- The outer function be
$$f(u) = e^u, \text{ so } f'(u) = e^u$$
- The inner function be
$$g(x) = x^3 + 1, \text{ so } g'(x) = 3x^2$$

Applying the Chain Rule, we then get:

$$\begin{aligned} h'(x) &= f'(g(x)) \cdot g'(x) \\ &= e^{x^3+1} \cdot 3x^2 \end{aligned}$$

Exercise Round 3

- The Chain Rule

Find the derivative of each function. Identify the inner and outer functions before applying the chain rule. For the first function, also state its real domain and differentiate on the interior of that domain.

1. $h(x) = (x^2 + x)^{\frac{3}{2}}$
2. $h(x) = \dfrac{1}{(3x^2 + 1)^2}$

Derivatives & Local Extrema

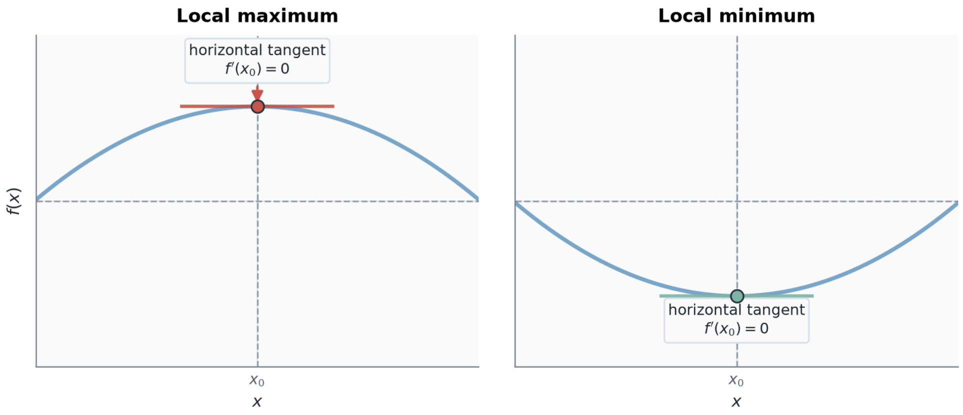
The derivative $f'(x)$ is the slope of the tangent line at $(x, f(x))$.

At a differentiable local maximum or minimum inside the domain, the tangent line is horizontal, so

$$f'(x) = 0.$$

An interior input can also be critical when $f'(x)$ is undefined. The corresponding graph point has no horizontal tangent line, but it may still be a local extremum.

Thus, interior inputs x for which $f'(x) = 0$ or $f'(x)$ is undefined are called critical numbers; the corresponding graph points $(x, f(x))$ are critical points.



To classify critical numbers, we can:

- Apply the first derivative sign test.
- Apply the second derivative test.

Classifying Local Extrema

- The First Derivative Sign Test

A critical number c is a candidate for a local extremum. Classify it by comparing the sign of f' on either side:

Left of c	**Right of c**	**Classification**
$f' > 0$	$f' < 0$	Local maximum
$f' < 0$	$f' > 0$	Local minimum
No sign change	No sign change	Neither

A practical workflow is:

1. Compute f'.
2. Find the critical numbers.
3. Check the sign of f' on both sides of each one.
4. Classify each critical number using the table.
5. Evaluate $f(c)$ if coordinates are needed.

When $f'(c)$ is undefined, the sign test still applies if:

- $f(c)$ exists, and
- the sign of f' can be checked on both sides.

Classifying Local Extrema

- The Second Derivative Test

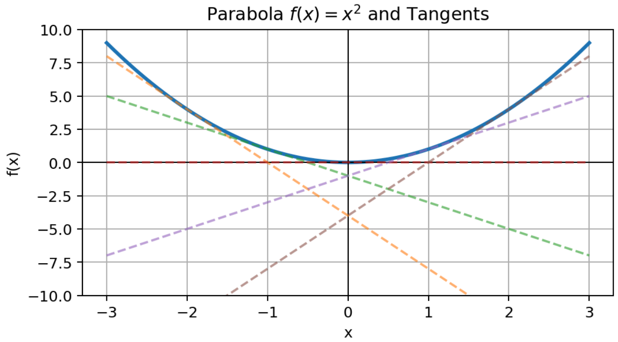
The second derivative of a function tells us how the slope is changing near critical numbers.

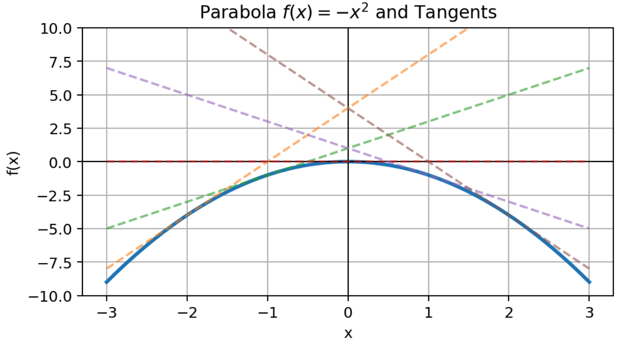
If $f'(c) = 0$ and $f''(c)$ exists, then:

- If $f''(c) > 0$: local minimum
- If $f''(c) < 0$: local maximum
- If $f''(c) = 0$: the test is inconclusive
- If $f''(c)$ does not exist: the test does not apply

For example, $x = 0$ is a critical number of:

$$\begin{aligned} f_1(x) &= x^2 & &\rightsquigarrow & f_1'(x) &= 2x & &\rightsquigarrow & f_1''(x) &= 2 \\ f_2(x) &= -x^2 & &\rightsquigarrow & f_2'(x) &= -2x & &\rightsquigarrow & f_2''(x) &= -2 \end{aligned}$$





Finding Local Extrema

- Example: A Cubic Function

Consider the function

$$f(x) = 2x^3 + \frac{3}{4}x^2 - \frac{5}{2}x.$$

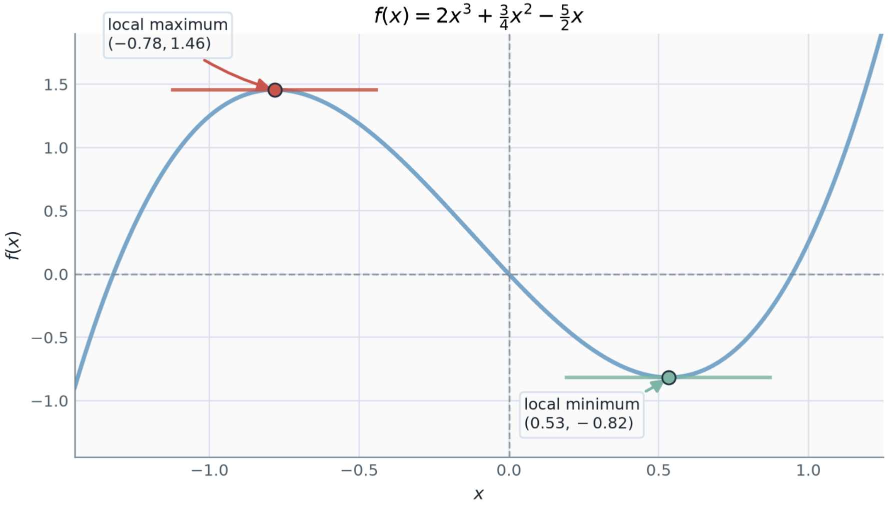
Find the critical numbers:

$$f'(x) = 6x^2 + \frac{3}{2}x - \frac{5}{2} = 0,$$

which gives $x_1 \approx -0.78$ and $x_2 \approx 0.53$.

Since $f''(x) = 12x + \frac{3}{2}$:

- $f''(x_1) < 0$: local maximum at $(-0.78, 1.46)$
- $f''(x_2) > 0$: local minimum at $(0.53, -0.82)$



Exercise Round 4

- Critical Numbers and Local Extrema

Find the local maxima or minima of the functions. In particular, find the critical numbers.

1. $f(x) = 2x^2 + 2x - 3$
2. $g(x) = 3x^3 + \frac{1}{4}x^2 - \frac{3}{2}x$