Mathematics Brush-Up for Data Science

Lecture Slides 1: Sets and Number Sets

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Set Theory

In maths we study:

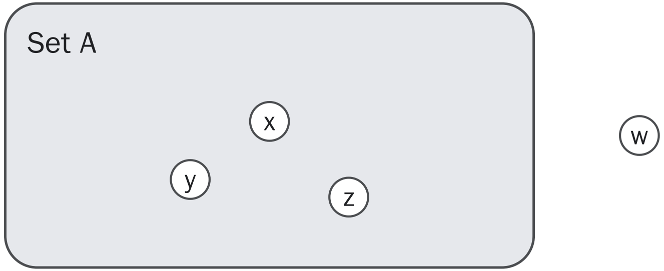
- Patterns, structures, quantities, and logical reasoning

Set theory:

- Is the foundational language of mathematics
- It provides a framework for discussing collections of objects

$\rightsquigarrow$ Understanding sets is crucial

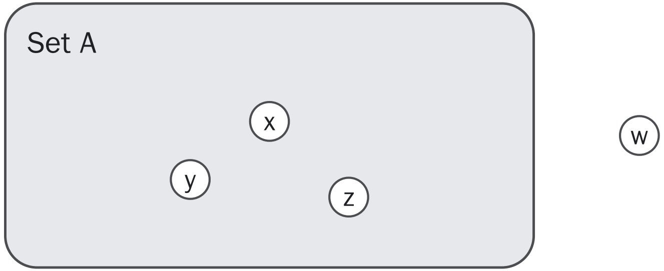
The Definition of a Set



A set is a well-defined collection of distinct objects, called elements.

The Definition of a Set

- Notation



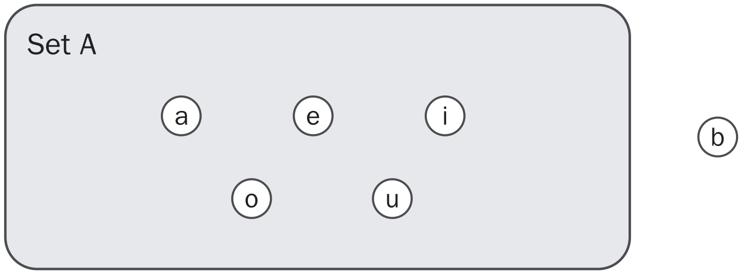
Sets are:

- Typically written using curly braces $\{\}$
- Denoted by capital letters from the Latin alphabet, such as $A = \{x, y, z\}$.

The elements x, y, and z are placeholders and can represent anything from numbers to abstract concepts, as long as they are clearly identifiable.

The Definition of a Set

- Example: Set Notation



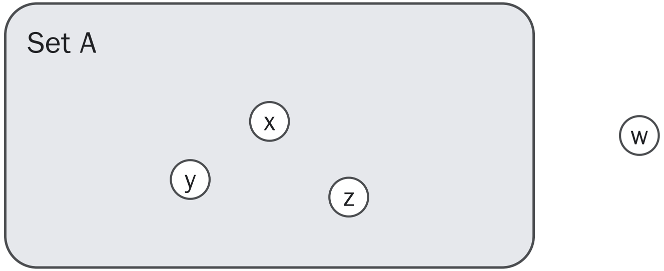
Consider the set of vowels in the English alphabet:

$$A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$$

This set clearly lists all the vowels, and it is easy to determine whether a given letter is a vowel or not.

The Definition of a Set

- Being Well-Defined



The objects inside a set, i.e., its elements, must be well-defined, meaning it is always clear whether something belongs to the set or not.

The Definition of a Set

- Example: Well-Defined Sets

For a set to be well-defined, it must be clear whether any given object is an element of the set or not.

The set of vowels in the word "radio" is well-defined and can be written as:

$$A = \{\mathrm{a}, \mathrm{i}, \mathrm{o}\}$$

Similarly, the following set would also be well-defined:

"All dates in 2025 on which weather station A recorded a daily minimum below 0° *C"*

because it is based on objective, measurable data.

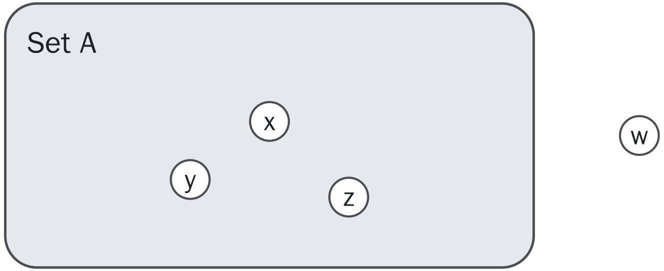
However, the following set would be ill-defined:

"All cold days last year"

because the term "cold" is subjective and can vary from person to person.

The Definition of a Set

- Distinct Elements



A set is determined by its distinct elements.

Repeating an element when listing a set:

- Is valid, but redundant
- Does not create another element

The Definition of a Set

- Example: Repeated Elements

Suppose we are given the following expression containing vowels of the English alphabet:

$$A = \{\mathrm{a}, \mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$$

The repeated entry is redundant: it does not create an additional element.

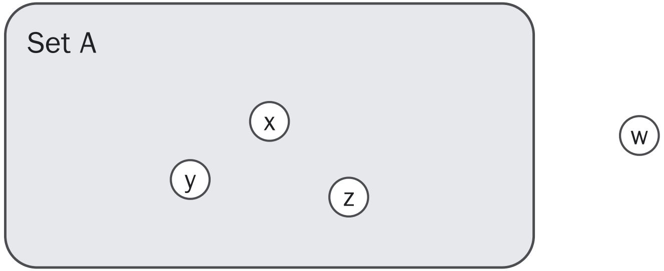
A clearer way to write the same set is:

$$A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$$

Both expressions describe the same set; the second simply lists each element once.

The Definition of a Set

- Order Independence



The order of elements in a set does not matter.

The Definition of a Set

- Example: Order Independence

Consider two sets containing the vowels of the English alphabet:

$$A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\} \quad \text{and} \quad B = \{\mathrm{u}, \mathrm{o}, \mathrm{i}, \mathrm{e}, \mathrm{a}\}$$

These two sets are identical because they contain the same elements.

In fact, we can write:

$$A = B$$

The order in which elements are listed does not affect the set being described.

The Empty Set

$$\emptyset = \{ \quad \}$$

The empty set:

- Is the unique set that contains no elements
- It plays a key role in set theory, similar to how 0 plays a key role in arithmetic

Common Number Sets

- Symbols & Meaning

Symbol	Name	Meaning and examples
$\mathbb{N}$	Natural numbers	The counting numbers: $1, 2, 3, \ldots$
$\mathbb{N}_0$	Natural numbers with zero	$0, 1, 2, 3, \ldots$
$\mathbb{Z}$	Integers	$\ldots, -2, -1, 0, 1, 2, \ldots$
$\mathbb{Q}$	Rational numbers	Numbers $\frac{p}{q}$, where p, q are integers and q is nonzero
$\mathbb{R}$	Real numbers	All numbers represented by points on the number line
$\mathbb{I}$	Irrational numbers	Real numbers that are not rational, such as $\sqrt{2}$ or π
$\mathbb{C}$	Complex numbers	Numbers $a + bi$, where a, b are real and $i^2 = -1$

The symbol $\mathbb{I}$ is used here for irrational numbers, but this notation is not universal.

Finite and Infinite Sets

A finite set:

- Has exactly n elements for some non-negative integer n
- Can be listed completely without repetition

For example, $\{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$ is finite. The empty set is also finite because it has zero elements.

An infinite set:

- Is not finite
- Has no finite list that contains all of its elements

For example, $\mathbb{N}$ and $\mathbb{R}$ are infinite.

Representing Sets

Sets can be described in different ways, depending on the context.

Choose a notation according to whether the elements:

- Can be listed conveniently
- Are best described by a condition
- Form an interval of real numbers

We will look at some common methods of representing sets, to understand:

- Their typical use-cases
- Their advantages

Representing Sets

- Verbal Description

Definition:
A verbal description defines a set using written language, explaining its elements or properties.

When to use:

- Providing context before using formal notation

Examples:

1. *"The set of vowels in the English alphabet."*
2. *"The set of non-negative integers."*
3. *"The set of non-negative integers strictly smaller than 6."*

Representing Sets

- Roster Form

Definition:
Roster form writes the elements of a set inside curly braces $\{\ \}$.

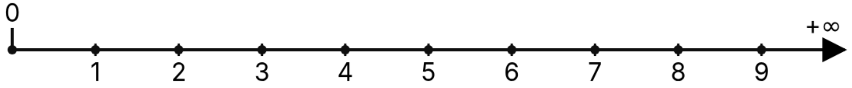
For an infinite set, an ellipsis may indicate an unambiguous continuation.

When to use:

- Finite sets
- Infinite sets with a clear, continuing pattern

Examples:

1. $A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$ — the English vowels
2. $B = \{0, 1, 2, 3, \ldots\}$ — the non-negative integers
3. $C = \{0, 1, 2, 3, 4, 5\}$ — the non-negative integers below 6



Representing Sets

- Set-Builder Notation

Definition:

A precise way to define a set by describing the rule its elements must satisfy.

$$\{x \mid \text{condition on } x\} \;\text{ or }\; \{x : \text{condition on } x\}$$

Both forms are read as “the set of all x such that the condition holds.”

- x is a placeholder, and the condition specifies which objects are included
- The bar ($\mid$) or colon ($:$) is read as “such that”

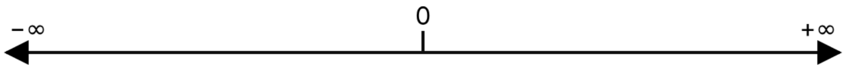
Examples:

1. $A = \{x \mid x \text{ is a vowel in the English alphabet}\}$
2. $B = \{x \mid x \text{ is a non-negative integer}\}$
3. $C = \{x \mid x \text{ is a non-negative integer and } x < 6\}$

Representing Sets

- The Real Numbers & Interval Notation

Certain sets appear so frequently in mathematics that they are assigned dedicated symbols. One of the most fundamental is the set of real numbers, denoted by $\mathbb{R}$.



- The real numbers can be represented by an infinite line, with one real number at each point
- Intervals are contiguous portions of that line

Interval notation:

- **[ ]** — inclusive bounds
- **( )** — exclusive bounds
- $\mathbb{R} = (-\infty, \infty)$
- Infinity is never included, so parentheses are always used beside $\pm\infty$

Representing Sets

- Examples: Interval Notation

In the set-builder descriptions below, x denotes a real number.

1. The real numbers strictly between 0 and 1:

$$(0, 1) = \{x \mid 0 < x < 1\}$$

2. The real numbers between 2 and 5, including both:

$$[2, 5] = \{x \mid 2 \leq x \leq 5\}$$

3. The real numbers greater than 3:

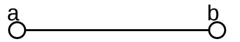
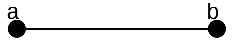
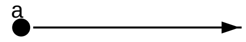
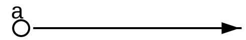
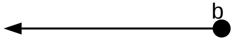
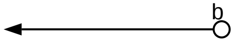
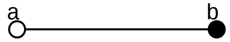
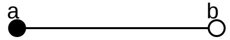
$$(3, \infty) = \{x \mid x > 3\}$$

4. The real numbers less than or equal to 0:

$$(-\infty, 0] = \{x \mid x \leq 0\}$$

Representing Sets

- The Real Numbers & Interval Notation (Continued)

Set	Interval Notation	Set-Builder Notation	Illustration
Open interval	(a, b)	$\{x \mid a < x < b\}$	
Closed interval	$[a, b]$	$\{x \mid a \leq x \leq b\}$	
Infinite to the right	$[a, \infty)$	$\{x \mid x \geq a\}$	
Infinite to the right	(a, ∞)	$\{x \mid x > a\}$	
Infinite to the left	$(-\infty, b]$	$\{x \mid x \leq b\}$	
Infinite to the left	$(-\infty, b)$	$\{x \mid x < b\}$	
Half-open (left open)	$(a, b]$	$\{x \mid a < x \leq b\}$	
Half-open (right open)	$[a, b)$	$\{x \mid a \leq x < b\}$	

Exercise Round 1

- Roster Form

Write the following sets in roster form (if possible). If it is not possible, explain why.

1. The set of the first five positive even integers.

2. The integers in the interval $(2, 7)$.

3. The set of non-negative integers that are considered very small.

4. The set of letters in the word "banana".

5. The set of all even integers.

6. The real numbers in the interval $[2, 2]$.

Exercise Round 1

- Set-Builder Notation

Use set-builder notation to describe the following sets.

7. $\{1, 2, 3, 4, 5, 6, 7\}$

8. $\{1, 10, 100, 1000, 10000\}$

9. $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$

10. $[7, 7]$, $(7, 7)$, $(7, 7]$, $[7, 7)$

Exercise Round 1

- Interval Notation

Use interval notation to describe the following sets.

11. The real numbers from -3, included, to 2, excluded.

12. The set of all real numbers strictly greater than -1.

13. The set of all real numbers less than or equal to 4.

14. The set of all real numbers greater than 0 and less than or equal to 10.

Set Relations

Understanding how elements relate to sets is fundamental, not only when defining a single set, but also when comparing and working with multiple sets.

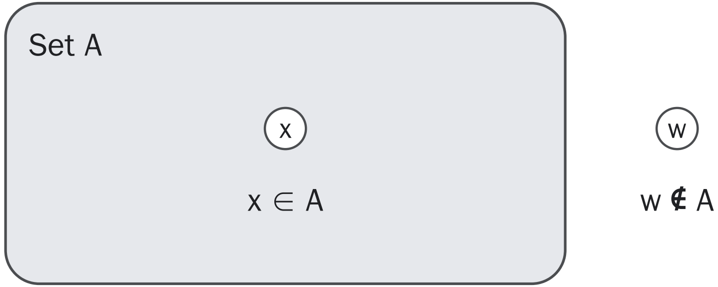
In this section, we introduce the following key relations and their common symbols:

- **Membership** ($\in$, $\notin$)
- **Equality** ($=$, $\neq$)
- **Cardinality** ($|A|$)
- **Subsets** ($\subseteq$, $\subset$)

These relations help describe and quantify how sets compare to each other in terms of their elements.

Set Membership

- Definition



Let x and w be objects and let A be a set. Then:

- $x \in A$ means that x is a member of A
- $w \notin A$ means that w is not a member of A

Set Membership

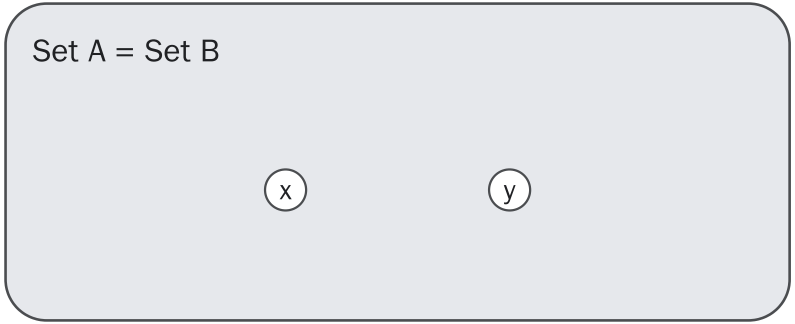
- Examples

The concept of set membership lets us concisely assess statements such as:

1. Is $\frac{7}{4} \in [1, 2)$?
2. Is $\{1\} \in A$ when $A = \{1, 2, 3\}$?
3. Is $\{1\} \in B$ when $B = \{\{1\}, \{2\}, \{3\}\}$?

Equality of Sets

- Definition



Two sets A and B are equal ($A = B$) if they contain exactly the same elements:

- Every element of A is in B
- Every element of B is in A

For example, if $A = \{x, y\}$ and $B = \{x, y\}$, we can check that $x, y \in A$ and $x, y \in B$, meaning $A = B$.

Cardinality

- Definition & Examples

Definition

The cardinality of a set A, written $|A|$, indicates the number of elements in A.

Examples:

1. Let $A = \{\mathrm{a}, \mathrm{e}, \mathrm{i}, \mathrm{o}, \mathrm{u}\}$, then $|A| = 5$.
2. The empty set $\emptyset$ has cardinality 0.
3. Let $S = \{1, 2, 3\}$ and $T = \{5, 3, 8\}$, then $|S| = |T| = 3$ even though $S \neq T$.

Set Relations

- Equality vs. Same Cardinality

Equality ($A = B$):

- Both sets have exactly the same elements.

Same cardinality ($|A| = |B|$):

- Sets have the same number of elements (same cardinality).

Key points:

- If $A = B$, then it must be the case that $|A| = |B|$.
- If $|A| = |B|$, it does not follow that $A = B$.

Exercise Round 2

- Equality & Cardinality

For each pair of sets:

- determine whether $P = Q$;
- compute $|P|$ and $|Q|$;
- state whether the sets have the same cardinality.

1. Let $P = \{1, 2, 3, 4\},\ Q = \{3, 2, 1, 4\}$.

2. Let $P = \{1, 2, 3\},\ Q = \{a, b, c\}$.

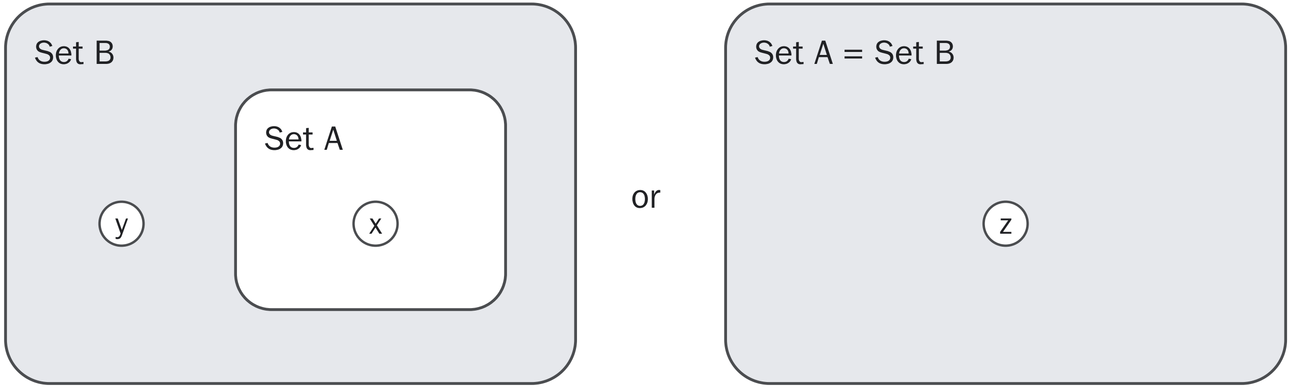
3. Let $P = \{\{1, 2\}, \{1, 3\}\},\ Q = \{\{1, 2\}, \{3, 1\}\}$.

4. Let $P = \{\},\ Q = \{\emptyset\}$.

5. Let $P = \{\{1, 2\}\},\ Q = \{\{1, 2\}, \{2, 1\}\}$.

A Subset

- Definition



A is a subset of B ($A \subseteq B$) if every element of A is in B.

For example, let $A = \{x\}$ and $B = \{x, y\}$. Because x is the only element of A and $x \in B$, every element of A is in B. Hence, $A \subseteq B$.

Here, $x \in B$ compares an element with a set, whereas $A \subseteq B$ compares one set with another.

A Subset

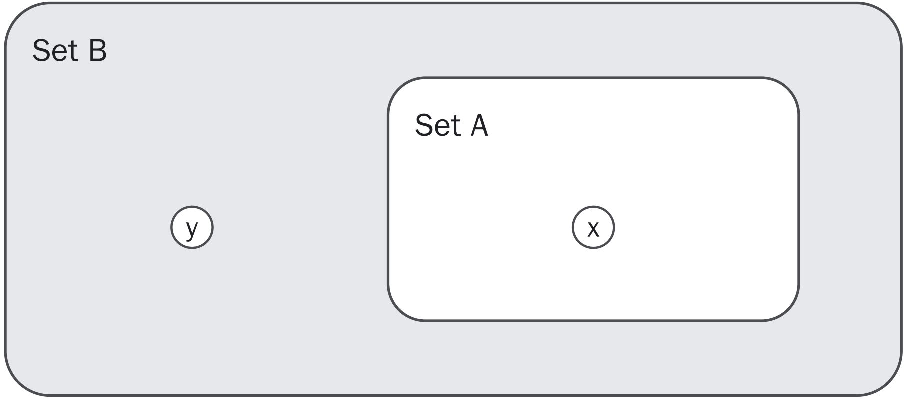
- Examples

In each case, determine whether S is a subset of T or vice versa:

1. Let $S = \{3, 5, 8\}$ and $T = \{5, 3, 8\}$
2. Let $S = \{\text{red}, \text{blue}\}$ and $T = \{\text{red}, \text{blue}, \text{green}\}$
3. Let $S = \emptyset$ and $T = \{5, 3, 8\}$
4. $S = \{3, 5, 8\}$ and $T = \{5, 8, 3, 2, 6\}$

A Proper Subset

- Definition



A is a proper subset of B ($A \subset B$) if:

- $A \subseteq B$, and
- $A \neq B$ (at least one element of B is not in A)

For example, if $A = \{x\}$ and $B = \{x, y\}$, then $A \subseteq B$ and $A \neq B$. Hence, $A \subset B$.

A Proper Subset

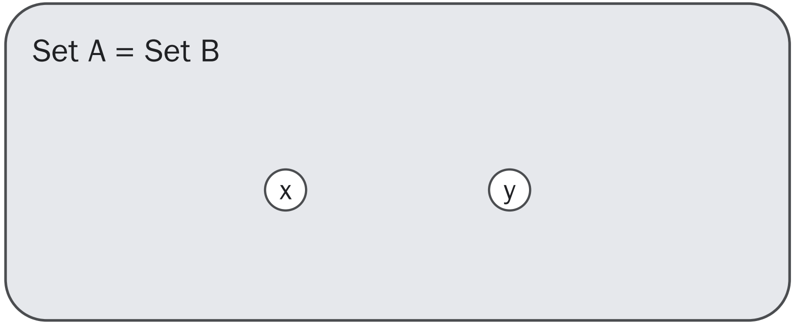
- Examples

In each case, determine whether S is a proper subset of T or vice versa:

1. Let $S = \{3, 5, 8\}$ and $T = \{5, 3, 8\}$
2. Let $S = \{\text{red}, \text{blue}\}$ and $T = \{\text{red}, \text{blue}, \text{green}\}$
3. Let $S = \emptyset$ and $T = \{5, 3, 8\}$
4. $S = \{3, 5, 8\}$ and $T = \{5, 8, 3, 2, 6\}$

Equality & Subsets

- Useful Consequence



Two sets are equal exactly when each is a subset of the other:

$$A = B \quad \text{exactly when} \quad A \subseteq B \text{ and } B \subseteq A$$

Each inclusion checks membership in one direction.

The Empty Set

- Useful Consequences

Two useful consequences distinguish the empty set from a set that contains it.

The empty set is a subset of every set A:

$$\emptyset \subseteq A$$

The empty set and a set containing the empty set are different:

$$|\emptyset| = 0, \qquad |\{\emptyset\}| = 1, \qquad \emptyset \neq \{\emptyset\}$$

Exercise Round 3

- Membership & Subsets

Let

$$X = \{0, 1, 2\}, \qquad Y = \{1, 2\}, \qquad Z = \{1, 2, 3\}.$$

State whether each expression is true or false.

1. $1 \in X$

2. $\{1\} \in X$

3. $Y \subseteq X$

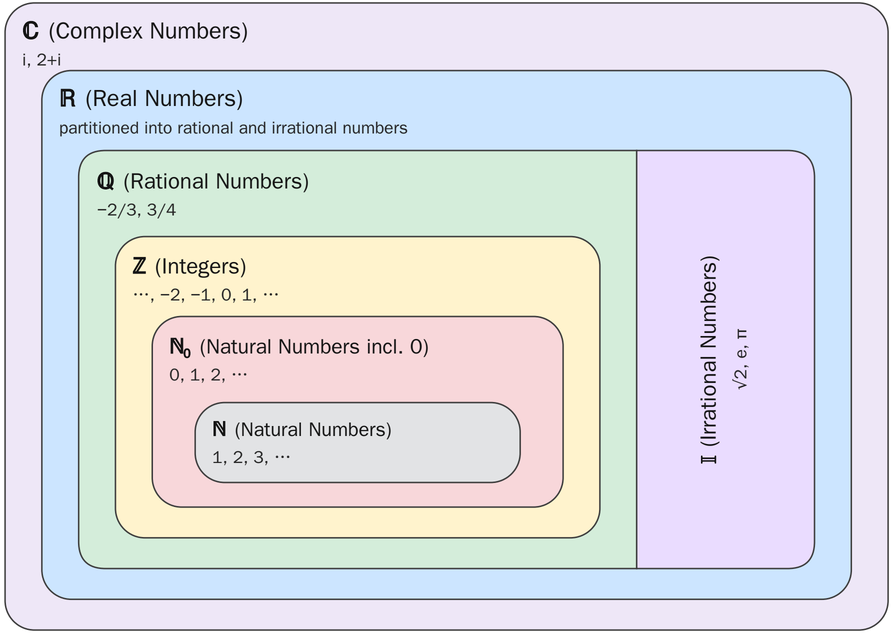
4. $X \subseteq Z$

5. $\emptyset \in Z$

6. $\emptyset \subseteq Z$

Number Sets

- A Hierarchy



- Common number sets form a nested hierarchy: $\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
- The irrational numbers $\mathbb{I}$ are real numbers that are not rational.

Exercise Round 4

- Number Sets

Complete the following tasks using the common number sets introduced above.

1. List all elements of the set

$$\{x \in \mathbb{N}_0 \mid x \leq 5\}.$$

2. Classify each number into the most specific applicable set from $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{I}$, and $\mathbb{C}$:

- -3
- 0
- $\frac{2}{5}$
- $\sqrt{2}$
- $4 + i$
- 1.33

3. List all elements of $\{x \in \mathbb{Z} \mid -2 \leq x \leq 2\}$

4. List all elements of $\{x \in \mathbb{Z} \mid x = x + 1\}$.